Finding Partial Derivatives

- The partial derivative with respect to *x* is just the <u>ordinary</u> derivative of the function *g* of a <u>single</u> variable that we get by keeping *y* fixed:

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example

- If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.
- Solution Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Solution (cont'd)

- Holding *x* constant and differentiating with respect to *y*, we get

$$f_x(x, y) = 3x^2y^2 - 4y$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Interpretations (cont'd)

- Partial derivatives can also be interpreted as rates of change.
- If $z = f(x, y)$, then…
  - $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed.
  - Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Example

- If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$.
- Interpret these numbers as slopes.
- Solution We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

Example

- If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
- <u>Solution</u> Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Example

- Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.
- <u>Solution</u> Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

- Similarly,

$$f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = e^{xy}/z$$

Higher Derivatives

- If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives

$$(f_x)_x, \ (f_x)_y, \ (f_y)_x, \text{ and } (f_y)_y,$$

which are called the *second partial derivatives* of f.

Higher Derivatives (cont'd)

- If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example

- Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

- Solution Earlier we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

- Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

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Mixed Partial Derivatives

- Note that $f_{xy} = f_{yx}$ in the preceding example, which is not just a coincidence.
- It turns out that $f_{xy} = f_{yx}$ for most functions that one meets in practice:

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Mixed Partial Derivatives (cont'd)

- Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x}$$

  and using Clairaut's Theorem we can show that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example

- Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.
- Solution

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$