# Approximations and Errors

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- The major advantage of numerical analysis is that a numerical answer can be obtained even when a problem has no "analytical" solution.

- Although the numerical technique yielded close estimates to the exact analytical solutions, there are errors because the numerical methods involve "approximations".

Approximations and Round-Off Errors

- For many engineering problems, we cannot obtain analytical solutions.
- Numerical methods yield approximate results, results that are close to the exact analytical solution.
  - Only rarely given data are exact, since they originate from measurements. Therefore there is probably error in the input information.
  - Algorithm itself usually introduces errors as well, e.g., unavoidable round-offs, etc …
  - The output information will then contain error from both of these sources.
- How confident we are in our approximate result?
- The question is "*how much error is present in our calculation and is it tolerable?*"

Accuracy and Precision

- **Accuracy** refers to how closely a computed or measured value agrees with the true value.

- **Precision** refers to how closely individual computed or measured values agree with each other.

- **Bias** refers to systematic deviation of values from the true value.

Increasing accuracy

Increasing precision

(a) (b) (c) (d)
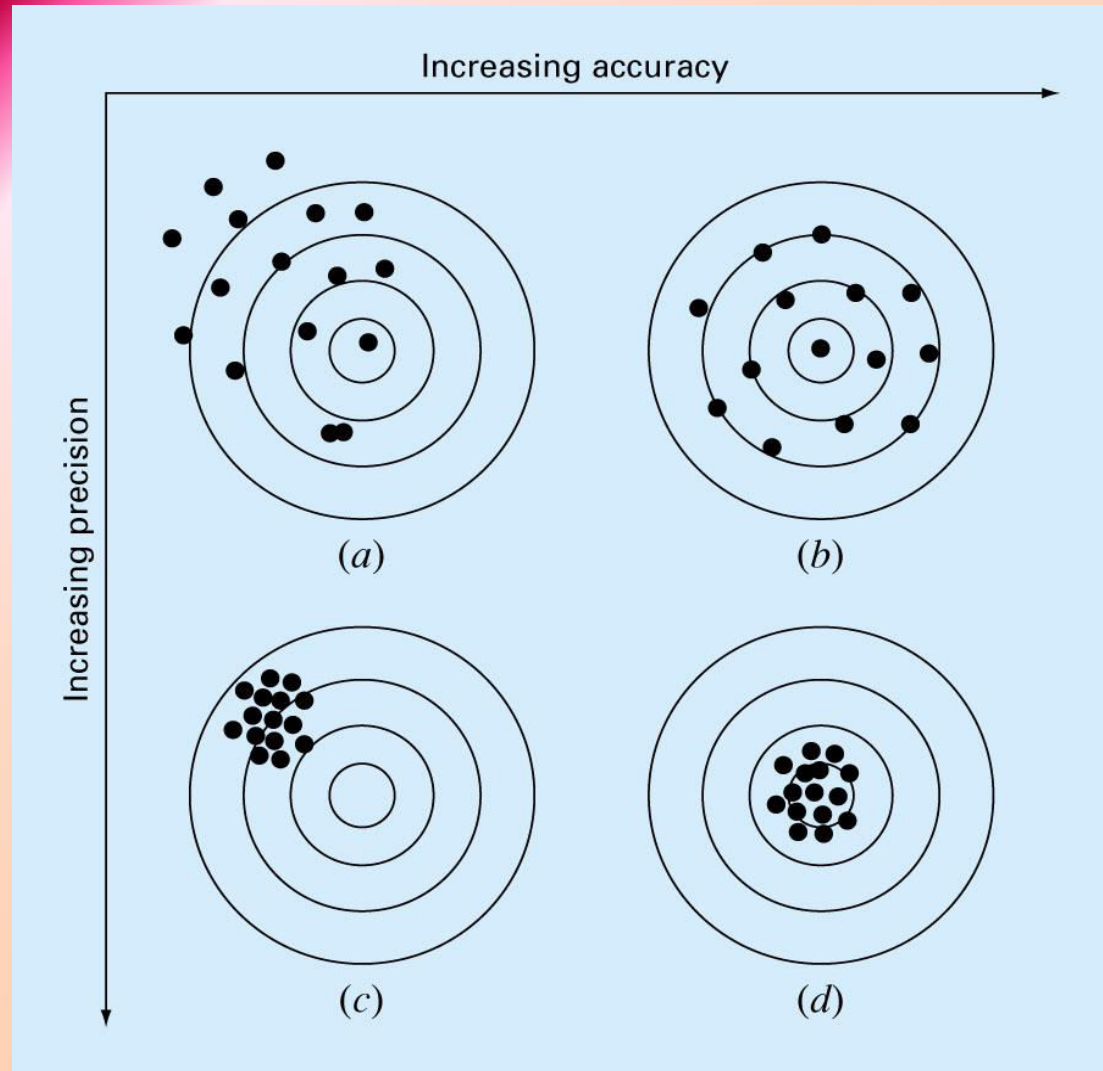
Significant Figures

Significant figures of a number are those that can be used with confidence.

Rules for identifying sig. figures:

- *All non-zero digits are considered significant*. For example, 91 has two significant digits (9 and 1), while 123.45 has five significant digits (1, 2, 3, 4 and 5).

- *Zeros appearing anywhere between two non-zero digits are significant*. Example: 101.12 has five significant digits.

- *Leading zeros are not significant*. For example, 0.00052 has two significant digits

- *Trailing zeros are generally considered as significant.* For example, 12.2300 has six significant digits.

Significant Figures

Scientific Notation

- If it is not clear how many, if any, of zeros are significant. This problem can be solved by using the scientific notation
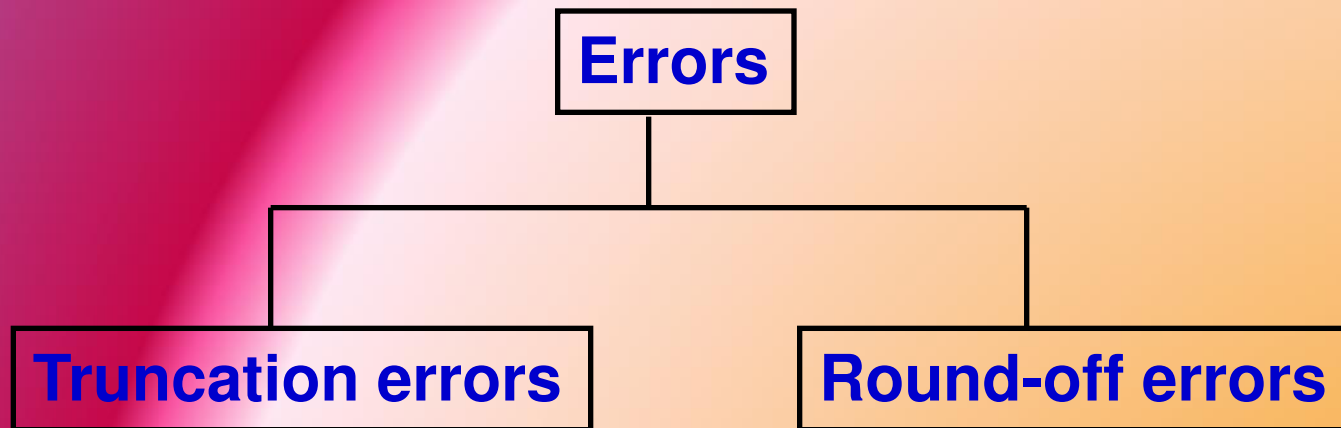
$$0.0013 = 1.3*10^{-3} \qquad 0.00130 = 1.30*10^{-3}$$

2 sig. figures   3 sig. figures

- If a number is expressed as $2.55 *10^4$, (3 s.f), then we are only confident about the first three digits. The exact number may be 25500, 25513, 25522.6 , .. etc. So we are not sure about the last two digits nor the fractional part- If any.

- However, if it is expressed as $2.550 * 10^4$, (4 s.f), then we are confident about the first four digits but uncertain about the last one and the fractional part – if any.

Error Definition

Numerical errors arise from the use of approximations

```
                    ┌─────────────┐
                    │   Errors    │
                    └──────┬──────┘
            ┌──────────────┴──────────────┐
┌───────────────────────┐    ┌───────────────────────┐
│  Truncation errors     │    │   Round-off errors     │
└───────────────────────┘    └───────────────────────┘
```

Result when approximations are used to represent exact mathematical procedure.

Result when numbers having limited significant figures are used to represent exact numbers.

Round-off Errors

- Numbers such as π, e, or $\sqrt{7}$ cannot be expressed by a fixed number of significant figures.

- Computers use a base-2 representation, they cannot precisely represent certain exact base-10 numbers

- Fractional quantities are typically represented in computer using "floating point" form, e.g.,

Example:

$\pi = 3.14159265358$ to be stored carrying 7 significant digits.

$\pi = 3.141592$ chopping

$\pi = 3.141593$ rounding

Truncation Errors

- Truncation errors are those that result using approximation in place of an exact mathematical procedure.

$$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{V\left(t_{i+1}\right) - V\left(t_i\right)}{t_{i+1} - t_i}$$

True Error

❑ **True error (E_t)**

> True error (E_t) or Exact value of error
> = true value – approximated value

❑ **True percent relative error (ε_t)**

$$True\ percent\ relative\ error = \varepsilon_t = \frac{True\ error}{True\ value} \times 100\ (\%)$$

$$= \frac{true\ value - approximated\ value}{true\ value} \times 100\ (\%)$$

Approximate Error

- The true error is known only when we deal with functions that can be solved analytically.

- In many applications, a prior true value is rarely available.

- For this situation, an alternative is to calculate an approximation of the error using the best available estimate of the true value as:

$$\varepsilon_a = Approximate\ percent\ relative\ error\ = \frac{Approximate\ error}{approximation} \times 100\ (\%)$$

Approximate Error

- In many numerical methods a present approximation is calculated using previous approximation:

$$\varepsilon_a = \frac{present\ approximation - previous\ approximation}{present\ approximation} \times 100\ (\%)$$

Note:

- The sign of ε_a or ε_t may be positive or negative
- We interested in whether the absolute value is lower than a prespecified tolerance (ε_s), not to the sign of error. Thus, the computation is repeated until (stopping criteria):

$$\left| \varepsilon_a \right| < \varepsilon_s$$

Prespecified Error

- We can relate ($\mathbf{\varepsilon}_s$) to the number of significant figures in the approximation,

  So, we can assure that the result is correct to at least n significant figures if the following criteria is met:

$$\varepsilon_s = (0.5 \times 10^{2-n}) \ \%$$

Example

The exponential function can be computed using Maclaurin series as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

Estimate $e^{0.5}$ using series, add terms until the absolute value of approximate error ε_a fall below a pre-specified error ε_s conforming with three significant figures.

{The exact value of $e^{0.5}=1.648721\ldots$}

- Solution

$$\varepsilon_s = \left(0.5 \times 10^{2-3}\right)\% = 0.05\%$$

✓ Using one term: $e^{0.5} = 1$ $\varepsilon_t = \frac{1.648721 - 1.0}{1.648721} 100\% = 39.3$

✓ Using two terms:

$e^{0.5} = 1 + 0.5 = 1.5$ $\varepsilon_t = \frac{1.648721 - 1.5}{1.648721} 100\% = 9.02\%$ $\varepsilon_a = \frac{1.5 - 1.0}{1.5} 100\% = 33.3\%$

✓ Using three terms:

$e^{0.5} = 1 + 0.5 + \frac{0.5^2}{2!} = 1.625$ $\varepsilon_t = \frac{1.648721 - 1.625}{1.648721} 100\% = 1.44\%$ $\varepsilon_a = \frac{1.625 - 1.0}{1.625} 100\% = 7.69\%$

Terms	Results	$\varepsilon_t\%$	$\varepsilon_a\%$
1	1.0	39.3	---
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158