

The Jacobian, which turns out to be a 2 x 2 determinant, is a natural outgrowth of the changing of variables in an integral problem.

In a single integral you can change variables by letting  $x = g(u)$ ,  $dx = g'(u)du$  and obtain

$$\int_a^b f(x)dx = \int_c^d g(u)g'(u)du \quad \text{with } a = g(c), \text{ and } b = g(d)$$

It's the double integral where we see the Jacobian.

$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$



This is the Jacobian

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

Let's work out the Jacobian where we do the conversion of rectangular to polar coordinates.

$$x = r \cos \theta \quad y = r \sin \theta$$

$$\frac{\partial x}{\partial r} = \cos \theta \quad \frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

*Hence,*

$$\iint_R f(x, y) dA = \iint_S f(r \cos \theta, r \sin \theta) r dr d\theta$$