- How to represent certain types of functions as sums of power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- You might wonder why we would ever want to express a known function as a sum of infinitely many terms.

  - Integration. (Easy to integrate polynomials) $\int e^{x^2} dx$
  - Finding limit $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$
  - Finding a sum of a series (not only geometric, telescoping) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

Example: $f(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots\cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

Example: Find Maclaurin series

$f(x) = \cos x$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x| < 1$$

MEMORIZE: these Maclaurin Series

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

Example:

Find Maclaurin series

$$f(x) = \tan^{-1} x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

The Maclaurin series for $f(x) = e^{-x^2/3}$ is given by

(a) $\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{3^n \cdot n!}$

(b) $\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{3 \cdot n!}$

(c) $\sum_{n=1}^{+\infty} (-1)^n \frac{x^{n}}{3^n \cdot n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(d) $\sum_{n=0}^{+\infty} \frac{x^{2n}}{3^n \cdot n!}$

(e) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n}}{9^n \cdot n!}$

The coefficient of x^{10} in the Maclaurin series of $f(x) = \sin(x^2)$ is equal to

(a) $\frac{1}{6}$

(b) 0

(c) $\frac{-1}{6}$

(d) $\frac{1}{120}$

(e) $\frac{1}{10}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The Maclaurin series for $f(x) = x^2 \cos(\sqrt{2}\, x)$ is

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^{2n+2}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{2})^n}{(2n)!} x^{2n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(\sqrt{2})^n}{(2n+1)!} x^{2n+1}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{2n} x^{2n+1}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{2}x)^{2n+2}}{(2n)!}$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The coefficient of x^4 in the Maclaurin series of $\cos^2 x$ is

(a) $\frac{1}{4}$

(b) $\frac{2}{3}$

(c) 2

(d) $\frac{1}{2}$

(e) $\frac{1}{3}$

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

Example:

Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

DEF:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

Example:

$$\binom{1/3}{3} = \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!} = \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{6} = \frac{5}{81}$$

Example:

$$\binom{1/2}{5} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)(\frac{1}{2}-4)}{5!}$$

binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

NOTE:

$$\binom{k}{0} = 1 \qquad \binom{k}{1} = \frac{k}{1!} = k \qquad \binom{k}{2} = \frac{k}{2!} = \frac{k(k-1)}{2!}$$

The Binomial Series

Using the binomial series, we get $\sqrt[3]{1+x} =$ (for $|x| < 1$)

(a) $1 + \frac{1}{3}\,x + \frac{1}{9}\,x^2 + \frac{1}{27}\,x^3 + \cdots$

(b) $1 + \frac{1}{3}\,x - \frac{1}{9}\,x^2 + \frac{5}{81}\,x^3 + \cdots$

(c) $1 + x - \frac{1}{2}\,x^2 + \frac{1}{6}\,x^3 + \cdots$

(d) $1 - \frac{1}{3}\,x + \frac{1}{6}\,x^2 + \frac{3}{27}\,x^3 + \cdots$

(e) $1 + \frac{1}{3}\,x + \frac{1}{9}\,x^2 - \frac{5}{81}\,x^3 + \cdots$

binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

If the Maclaurin series of $(1+x)^{3/2}$ is

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots,$$

then $D + E =$

(a) $-\frac{5}{128}$

(b) $\frac{9}{128}$

(c) $\frac{7}{16}$

(d) $-\frac{7}{16}$

(e) $-\frac{7}{128}$

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binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Important Maclaurin Series and Their Radii of Convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x| < 1$$

Example: Find Maclaurin series

$$f(x) = \ln(1-x)$$

Maclaurin series (center is 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

Taylor series (center is a)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots$$

The first three terms of the Taylor series of $f(x) = \cos(2x)$ about $a = \pi$ are given by

(a) $1 - 2(x-\pi)^2 + \frac{2}{3}(x-\pi)^4$

(b) $1 - 2(x-\pi) - 2(x-\pi)^2$

(c) $1 - 2(x-\pi)^2 + 16(x-\pi)^4$

(d) $-1 + 2(x-\pi) + \frac{4}{3}(x-\pi)^3$

(e) $1 + 2(x+\pi)^2 - \frac{2}{3}(x+\pi)^4$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

The first three nonzero terms of the Taylor series of $f(x) = \sin(2x)$ about $a = \frac{\pi}{2}$ are given by

(a) $-2\left(x-\frac{\pi}{2}\right)+\frac{4}{3}\left(x-\frac{\pi}{2}\right)^3-\frac{4}{15}\left(x-\frac{\pi}{2}\right)^5$

(b) $1-2\left(x-\frac{\pi}{2}\right)+\frac{4}{3}\left(x-\frac{\pi}{2}\right)^3$

(c) $-2\left(x-\frac{\pi}{2}\right)+\frac{4}{3}\left(x-\frac{\pi}{2}\right)^2-\frac{4}{15}\left(x-\frac{\pi}{2}\right)^3$

(d) $2\left(x-\frac{\pi}{2}\right)+\frac{4}{3}\left(x-\frac{\pi}{2}\right)^3-\frac{4}{15}\left(x-\frac{\pi}{2}\right)^5$

(e) $-2\left(x-\frac{\pi}{2}\right)+4\left(x-\frac{\pi}{2}\right)^2-4\left(x-\frac{\pi}{2}\right)^5$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

The Taylor series of $f(x) = \frac{1}{x}$ about $x = 2$ is

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^{n+1}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n}}(x-2)^{n}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^{n}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x+2)^{n}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n}}(x+2)^{n}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Taylor series (center is a)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

DEF:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor polynomial of order n

The Taylor polynomial of order 3 generated by the function f(x)=ln(3+x) at a=1 is:

(a) $\ln 4 + \frac{(x-1)}{4} - \frac{(x-1)^2}{32} + \frac{(x-1)^3}{192}$

(b) $\ln 4 - \frac{(x-1)}{4} + \frac{(x-1)^2}{32} - \frac{(x-1)^3}{192}$

(c) $\frac{(x-1)}{4} - \frac{(x-1)^2}{32} + \frac{(x-1)^3}{192} - \frac{(x-1)^4}{256}$

(d) $-\frac{(x-1)}{4} + \frac{(x-1)^2}{32} - \frac{(x-1)^3}{192} + \frac{(x-1)^4}{256}$

(e) $\ln 4 + \frac{(x-1)}{4} + \frac{(x-1)^2}{32} + \frac{(x-1)^3}{192}$

DEF:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor polynomial of order n

The first three terms of the Taylor series of $f(x) = \dfrac{1}{\sqrt{x}}$ about $a = 4$ are given by

(a) $\dfrac{1}{4} - \dfrac{1}{16}(x-4) + \dfrac{3}{4}(x-4)^2$

(b) $\dfrac{1}{2} - \dfrac{1}{2}(x-4) + \dfrac{3}{4}(x-4)^2$

(c) $\dfrac{1}{2} - (x-4) + (x-4)^2$

(d) $\dfrac{1}{2} + \dfrac{1}{16}(x+4) - \dfrac{1}{128}(x+4)^2$

(e) $\dfrac{1}{2} - \dfrac{1}{16}(x-4) + \dfrac{3}{256}(x-4)^2$

The first four terms of the Taylor series of $f(x) = 4 + \ln x$ about $a = 1$ are given by

(a) $4 + (x-1) - (x-1)^2 + 2(x-1)^3$

(b) $4 + (x+1) - (x+1)^2 + 2(x+1)^3$

(c) $4 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

(d) $4 + 5(x-1) - \frac{3}{2}(x-1)^2 + (x-1)^3$

(e) $4 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor series (center is a)

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor polynomial of order n

$$R_n(x)=\sum_{k=n+1}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Remainder

$$f(x)=P_n(x)+R_n(x)$$

Taylor Series

Taylor's Formula

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^k$$

Remainder consist of infinite terms

for some *c* between *a* and *x*.

REMARK: Observe that : $f^{(n+1)}(c)$ not $f^{(n+1)}(a)$

Taylor's Formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^k$$

for some *c* between *a* and *x*.

Taylor's Formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^k$$

for some *c* between *0* and *x*.

Taylor series (center is a)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

DEF:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

***n*th-degree Taylor polynomial of *f* at *a*.**

DEF:

$$R_n(x) = f(x) - T_n(x)$$

Remainder

Example:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$T_3(x) = \sum_{n=0}^{3} x^n = 1 + x + x^2 + x^3$$

$$R_3(x) = \sum_{n=4}^{\infty} x^n = x^4 + x^5 + x^6 + \cdots\cdots$$