# Characteristic Equatuion Eigenvalues and Eigenvectors

# Definitions

Definition : A nonzero vector **x** is an ***eigenvector*** (or *characteristic vector*) of a square matrix **A** if there exists a scalar **λ** such that **Ax = λx**. Then **λ** is an ***eigenvalue*** (or *characteristic value*) of **A**.

*Note*: The zero vector can not be an eigenvector even though **A0 = λ0**. But **λ = 0** can be an eigenvalue.

Example:

$$Show\ \ x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\ is\, an\, eigenvector\ \ for\ \ A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$$

$$Solution : Ax = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$But \quad for \quad \lambda = 0, \quad \lambda x = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Thus,\ x\ is\, an\, eigenvector\ of\ \ A, and\ \lambda = 0\ is\, an\, eigenvalue.$$

# Geometric interpretation of Eigenvalues and Eigenvectors

An $n \times n$ matrix **A** multiplied by $n \times 1$ vector **x** results in another $n \times 1$ vector **y=Ax**. Thus **A** can be considered as a transformation matrix.

In general, a matrix acts on a vector by changing both its magnitude and its direction. However, a matrix may act on certain vectors by changing only their magnitude, and leaving their direction unchanged (or possibly reversing it). These vectors are the **eigenvectors** of the matrix.

A matrix acts on an eigenvector by multiplying its magnitude by a factor, which is positive if its direction is unchanged and negative if its direction is reversed. This factor is the **eigenvalue** associated with that eigenvector.

# Eigenvalues

Let x be an eigenvector of the matrix **A.** Then there must exist an eigenvalue **λ** such that $\mathbf{Ax = \lambda x}$ or, equivalently,

$$\mathbf{Ax - \lambda x = 0} \quad \text{or}$$

$$\mathbf{(A - \lambda I)x = 0}$$

If we define a new matrix **B = A – λI**, then

$$\mathbf{Bx = 0}$$

If **B** has an inverse then $\mathbf{x = B^{-1}0 = 0}$. But an eigenvector cannot be zero.

Thus, it follows that x will be an eigenvector of **A** if and only if **B** does not have an inverse, or equivalently **det(B)=0**, or

$$\mathbf{\det(A - \lambda I) = 0}$$

This is called the **characteristic equation** of **A**. Its roots determine the eigenvalues of **A**.

# Eigenvalues: examples

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: −1, − 2

*Note:* The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

λ = 2 is an eigenvector of multiplicity 3.

# Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If $\boldsymbol{\lambda_i}$ is an eigenvalue then the corresponding eigenvector $\mathbf{x_i}$ is the solution of $(\mathbf{A} - \boldsymbol{\lambda_i}\mathbf{I})\mathbf{x_i} = \mathbf{0}$

## Example 1 (cont.):

$$\lambda = -1 : (-1)I - A = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda = -2 : (-2)I - A = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \end{bmatrix},\ s \neq 0$$

# Eigenvectors

Example 2 (cont.): Find the eigenvectors of $A=\begin{bmatrix}2&1&0\\0&2&0\\0&0&2\end{bmatrix}$

Recall that λ = 2 is an eigenvector of multiplicity 3.

Solve the homogeneous linear system represented by

$$(2I-A)\mathbf{x}=\begin{bmatrix}0&-1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Let $x_1=s,\ x_3=t$. The eigenvectors of λ = 2 are of the form

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}s\\0\\t\end{bmatrix}=s\begin{bmatrix}1\\0\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\end{bmatrix},$$

*s* and *t* not both zero.

# Properties of Eigenvalues and Eigenvectors

Definition: The trace of a matrix A, designated by tr(A), is the sum of the elements on the main diagonal.

Property 1: The sum of the eigenvalues of a matrix equals the trace of the matrix.

Property 2: A matrix is singular if and only if it has a zero eigenvalue.

Property 3: The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

Property 4: If **$\lambda$** is an eigenvalue of **A** and **A** is invertible, then **$1/\lambda$** is an eigenvalue of matrix **$A^{-1}$**.

# Properties of Eigenvalues and Eigenvectors

Property 5: If **λ** is an eigenvalue of **A** then **kλ** is an eigenvalue of **kA** where **k** is any arbitrary scalar.

Property 6: If **λ** is an eigenvalue of **A** then $\lambda^k$ is an eigenvalue of $A^k$ for any positive integer **k**.

Property 8: If **λ** is an eigenvalue of **A** then **λ** is an eigenvalue of $A^T$.

Property 9: The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.

# Linearly independent eigenvectors

**Theorem:** Eigenvectors corresponding to distinct (that is, different) eigenvalues are linearly independent.

**Theorem:** If **λ** is an eigenvalue of multiplicity **k** of an **n** $\times$ **n** matrix **A** then **the number of linearly independent** eigenvectors of **A** associated with **λ** is given by **m = n - r(A- λI)**. Furthermore, **1 ≤ m ≤ k**.

*Example 2 (cont.):* The eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

*s* and *t* not both zero.

$\lambda = 2$ has ***two linearly independent eigenvectors***