Inverse of a Matrix

Definition of Inverse

The inverse is Defined as $AA^{-1} = A^{-1}A = I$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We will find out how to calculate the inverse for 2x2 matrix

But first why is it important ?

Because it will allow us to solve equations of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

We will only consider 2x2 Matrix systems
That means simultaneous equations

Why will it help us solve equations?

Because if we can express a system of equations in the form

$$Ax = b$$

Then we can multiply both sides by the inverse matrix

$$A^{-1}Ax = A^{-1}b$$

And we can then know the values of X because $\boxed{A^{-1}A = I}$

$$\boxed{\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}}$$

Recap Multiplication

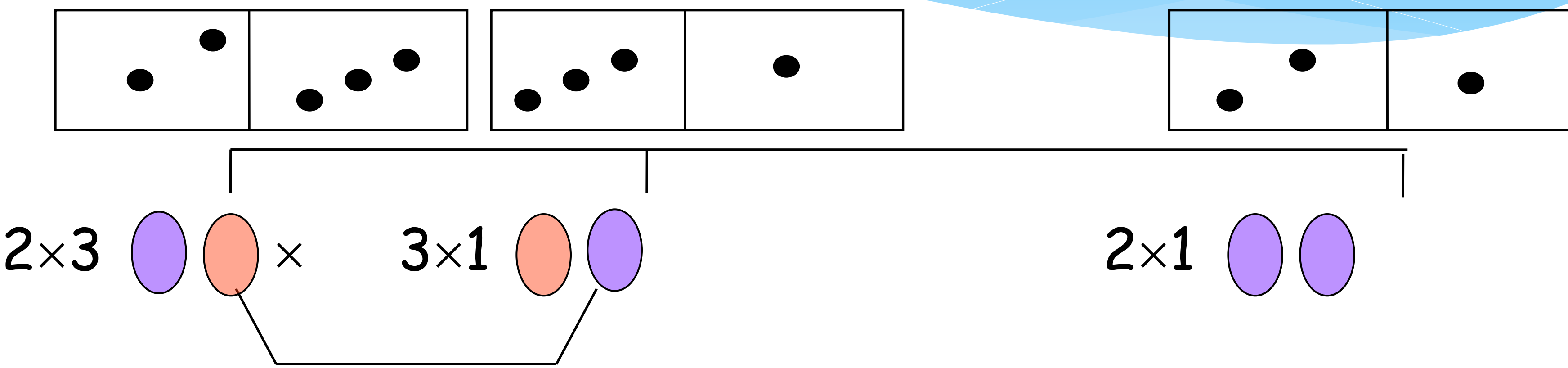
$$\begin{pmatrix} 4 & -3 & 2 \\ -6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 8+3+12 \\ -12+0+6 \end{pmatrix} = \begin{pmatrix} 23 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 3 \\ 1 & 2 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 12+9 \\ 2+6 \\ 10+21 \end{pmatrix} = \begin{pmatrix} 21 \\ 8 \\ 31 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 24+10 & 4+8 \end{pmatrix} = \begin{pmatrix} 34 & 12 \end{pmatrix}$$

Shapes and sizes

* Dimensions and compatibility given by the domino rule



To multiply the columns of the first must be equal to the rows of the second

The dimensions of the result are given by the 2 outer numbers

Note matrix multiplication is **not** commutative.

If A is a 3x1 and B is a 1x3 then AB is 3x3 BA is 1x1

The multiplicative inverse of a matrix

* This can only be done with SQUARE matrices
* By hand we will only do this for a 2x2 matrix
* Inverses of larger square matrices can be calculated but can be quite time expensive for large matrices, computers are generally used

Ex $\begin{pmatrix} 4 & 8 \\ 1 & 3 \end{pmatrix}$ A = then A^{-1} = $\begin{pmatrix} 0.75 & -2 \\ -0.25 & 1 \end{pmatrix}$ as AxA^{-1} = I

$$\begin{pmatrix} 4 & 8 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0.75 & -2 \\ -0.25 & 1 \end{pmatrix} = \begin{pmatrix} 3-2 & -8+8 \\ 0.75-0.75 & -2+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Finding the Inverse of a 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Step-1 First find what is called the Determinant

This is calculated as ad-bc

Step-2 Then swap the elements in the leading diagonal

$$\begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

Step-3 Then negate the other elements

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step-4 Then multiply the Matrix by 1/determinant

$$\frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example Find Inverse of A

Step 1 – Calc Determinant

$$A = \begin{pmatrix} 4 & 8 \\ 1 & 3 \end{pmatrix}$$

Determinant (ad-cb) = 4x3-8x1 = 4

Step 2 – Swap Elements on leading diagonal

step2 $\begin{pmatrix} 3 & 8 \\ 1 & 4 \end{pmatrix}$

Step 3 – negate the other elements

step3 $\begin{pmatrix} 3 & -8 \\ -1 & 4 \end{pmatrix}$

Step 4 – multiply by 1/determinant

step4 $\frac{1}{4}\begin{pmatrix} 3 & -8 \\ -1 & 4 \end{pmatrix}$

$$A^{-1} = \begin{bmatrix} 0.75 & -2 \\ -0.25 & 1 \end{bmatrix}$$

check

$$AA^{-1} = \begin{bmatrix} 4 & 8 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.75 & -2 \\ -0.25 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & -8+8 \\ 0.75-0.75 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the inverses and check them

$$A = \begin{pmatrix} 2 & 6 \\ 1 & 5 \end{pmatrix} \qquad A^{-1} = \frac{1}{4}\begin{pmatrix} 5 & -6 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1.25 & -1.5 \\ -0.25 & 0.5 \end{pmatrix}$$

$$B = \begin{pmatrix} -5 & 20 \\ -1 & 2 \end{pmatrix} \qquad B^{-1} = \frac{1}{10}\begin{pmatrix} 2 & -20 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 0.2 & -2 \\ 0.1 & -0.5 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix} \qquad C^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0.5 & 1 \\ 0 & -1 \end{pmatrix}$$

More inverses to find and check

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \qquad A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1.5 & -2 \\ -0.5 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 & 10 \\ -1 & 2 \end{pmatrix} \qquad B^{-1} = \frac{1}{20}\begin{pmatrix} 2 & -10 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0.1 & -0.5 \\ 0.05 & 0.25 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \qquad C^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 8 & 2 \\ -1 & 0 \end{pmatrix} \qquad D^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0.5 & 4 \end{pmatrix}$$

$$E = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix}$$

E^{-1} *cannot be found as* $\det = 2\times 4-(-1\times -8) = 8-8 = 0$

$$F = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F^{-1} = \frac{1}{1}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ self inverting}$$

Applications of matrices

* Because matrices are clever storage systems for numbers there are a large and diverse number of ways we can apply them.

* Matrices are used in to solve equations on computers

  - solving equations

* They are used in computer games and multi-media devices to move and change objects in space

  - transformation geometry

* We only consider solving equations on Maths1 with using 2x2 matrices

Solving simultaneous equations

- We can use our 2x2 matrices to express 2 simultaneous equations (2 equations about the same 2 variables)
- First we must put them in the correct format
- for the variables x & y the format should be

  ax + by = m

  cx + dy = n {where a,b,c,d,m & n are constants}

Example

Peter and Jane spend £240 altogether and Peter spends 3 times as much as Jane.

let p: what Peter spends and j: what Jane spends

then p + j = 240 (right format a and b = 1 m = 240)

p=3j (wrong format)

rewrite p-3j = 0 (right format c = 1 d = -3 n = 0)

Solving simultaneous equations

We can use our 2x2 matrices to express these simultaneous equations

$$\left.\begin{aligned} x + y &= 240 \\ x - 3y &= 0 \end{aligned}\right\}$$

Becomes in matrix form

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 240 \\ 0 \end{pmatrix}$$

constants from the left hand side

UNKNOWNS
X ~ x1
Y ~ x2

constants from the right hand side

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 240 \\ 0 \end{pmatrix}$$

Format is $Ax=B$

To solve this using the matrix we must get rid of it by using its inverse!

First find the inverse

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}^{-1} = \frac{1}{-4}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & -0.25 \end{pmatrix}$$

now use it on both sides of the equation

$$\begin{pmatrix} 0.75 & 0.25 \\ 0.25 & -0.25 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & -0.25 \end{pmatrix}\begin{pmatrix} 240 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & -0.25 \end{pmatrix}\begin{pmatrix} 240 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 180 \\ 60 \end{pmatrix}$$

So *Answer* is p = 180 j = 60

Summary of method

1. Format the simultaneous equations for variable x & y

$$ax + by = m$$
$$cx + dy = n$$

2. Rewrite them in matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$$

3. Find the inverse of the 2x2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

4. Solve for the variables x,y by multiplying the right hand side of the equation by the inverse

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix}$$

Solve the following

3x +4y = 5

5x = 7-6y

$$\begin{pmatrix}3 & 4\\5 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\7\end{pmatrix}$$

Answer x = -1 y = 2

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{-2}\begin{pmatrix}6 & -4\\-5 & 3\end{pmatrix}\begin{pmatrix}5\\7\end{pmatrix}=\frac{1}{-2}\begin{pmatrix}30-28\\-25+21\end{pmatrix}=\frac{1}{-2}\begin{pmatrix}2\\-4\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$$

x+7y = 1.24

3y -x = 0.76

$$\begin{pmatrix}1 & 7\\-1 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1.24\\0.76\end{pmatrix}$$

Answer x = -0.16 y = 0.2

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{10}\begin{pmatrix}3 & -7\\1 & 1\end{pmatrix}\begin{pmatrix}1.24\\0.76\end{pmatrix}=\frac{1}{10}\begin{pmatrix}3.72-5.32\\1.24+0.76\end{pmatrix}=\frac{1}{10}\begin{pmatrix}-1.6\\2\end{pmatrix}=\begin{pmatrix}-0.16\\0.2\end{pmatrix}$$

8x = 3y -1

x+y =-7

$$\begin{pmatrix}8 & -3\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\-7\end{pmatrix}$$

Answer x = -2 y = -5

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{11}\begin{pmatrix}1 & 3\\-1 & 8\end{pmatrix}\begin{pmatrix}-1\\-7\end{pmatrix}=\frac{1}{11}\begin{pmatrix}-1-21\\1-56\end{pmatrix}=\frac{1}{11}\begin{pmatrix}-22\\-55\end{pmatrix}=\begin{pmatrix}-2\\-5\end{pmatrix}$$