LINEAR TRANSFORMATIONS

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- V, W vector spaces with same fields F.
  - Definition: $T:V \rightarrow W$ s.t. $T(ca+b)=c(Ta)+Tb$ for all a,b in V. c in F. Then T is linear.
  - Property: $T(O)=O$. $T(ca+db)=cT(a)+dT(b)$, a,b in V, c,d in F. (equivalent to the def.)
  - Example: A mxn matrix over F. Define T by $Y=AX$. $T:F^n \rightarrow F^m$ is linear.
    - Proof: $T(aX+bY)= A(aX+bY)=aAX+bAY = aT(X)+bT(Y)$.

- $U:F^{1xm} \to F^{1xn}$ defined by $U(a)=aA$ is linear.
- Notation: $F^m=F^{mx1}$
- Remark: $L(F^{mx1},F^{nx1})$ is same as $M_{mxn}(F)$.
  - For each mxn matrix A we define a unique linear transformation Tgiven by $T(X)=AX$.
  - For each a linear transformation T has A such that $T(X)=AX$. We will discuss this in section 3.3.
  - Actually the two spaces are isomorphic as vector spaces.
  - If m=n, then compositions correspond to matrix multiplications exactly.

- Example: T(x)=x+4. F=R. V=R. This is not linear.
- Example: V = {f polynomial:F➔F}
  T:V ➔V defined by T(f)=Df.

- V={f:R➔R continuous}

$$\begin{aligned} f(x) &= c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k \\ Df(x) &= c_1 + 2c_2 x + \cdots + k c_k x^{k-1} \end{aligned}$$

$$Tf(x) = \int_0^x f(t)dt$$

- Null space of T :V➔W:= { v in V | Tv = 0}.
- Rank T:= dim{Tv | v in V} in W. = dim range T.
- Null space is a vector subspace of V.
- Range T is a vector subspace of W.
- Example:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Null space z=t=0. x+2y=0 dim =1
- Range = W. dim = 3

- Theorem: rank T + nullity T = dim V.
- Proof: $a_1,\dots,a_k$ basis of N. dim N = k. Extend to a basis of V: $a_1,\dots,a_k, a_{k+1},\dots,a_n$.
  - We show $T a_{k+1},\dots,T a_n$ is a basis of R. Thus n-k = dim R. n-k+k=n.
    - Spans R:
    - Independence:

$$
\begin{aligned}
v &= x_1\alpha_1 + \cdots + x_n\alpha_n \\
Tv &= x_{k+1}T(\alpha_{k+1}) + \cdots x_n T(\alpha_n)
\end{aligned}
$$

$$
\begin{aligned}
\textstyle\sum_{i=k+1}^{n} c_i T\alpha_i &= 0 \\
T(\textstyle\sum_{i=k+1}^{n} c_i\alpha_i) &= 0 \\
\textstyle\sum_{i=k+1}^{n} c_i\alpha_i &\in N \\
\textstyle\sum_{i=k+1}^{n} c_i\alpha_i &= \textstyle\sum_{i=1}^{k} c_i\alpha_i \\
c_i &= 0, i = k+1,\dots,n
\end{aligned}
$$

- Theorem 3: A mxn matrix.
  Row rank A = Column rank A.
- Proof:
  - column rank A = rank T where $T:R^n \rightarrow R^m$ is defined by $Y=AX$. e_i goes to i-th column. So range is spaned by column vectors.
  - rankT+nullityT=n by above theorem.
  - column rank A+ dim S = n where $S=\{X \mid AX=O\}$ is the null space.
  - dim S= n - row rank A (Example 15 Ch. 2 p.42)
  - row rank = column rank.

- A^{mxn}. S solution space. R r-r-e matrix
- r = number of nonzero rows of R.
- RX=0 $k_1<k_2<\ldots<k_r$. J= {1,..,n}- $\{k_1,k_2,\ldots,k_r\}$.

$$\begin{array}{lllllll} x_{k_1} & & & & + & \sum_{j=1}^{n-r} C_{1j}u_j & = 0 \\ & x_{k_2} & & & + & \sum_{j=1}^{n-r} C_{2j}u_j & = 0 \\ & & \ddots & & + & \vdots & = \vdots \\ & & & x_{k_r} & + & \sum_{j=1}^{n-r} C_{rj}u_j & = 0 \end{array}$$

  - Solution spaces parameter $u_1,\ldots,u_{n-r}$.
  - Or basis E_j given by setting $u_j=1$ and other $u_i=0$ and $x_{ki}=c_{ij}$.

ALGEBRA OF LINEAR TRANSFORMATIONS

- Linear transformations can be added, and multiplied by scalars. Hence they form a vector space themselves.
- Theorem 4: T,U:V➔W linear.
  - Define T+U:V➔W by (T+U)(a)=T(a)+U(a).
  - Define cT:V➔W by cT(a)=c(T(a)).
  - Then they are linear transformations.

- Definition: L(V,W)={T:V➔W | T is linear}.
- Theorem 5: L(V,W) is a finite dim vector space if so are V,W. dimL=dimVdimW.
- Proof: We find a basis:

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \subset V$$

  - Define a linear transformation V➔W:

$$\mathcal{B}' = \{\beta_1, \ldots, \beta_m\} \subset W$$

  - The basis:

$$E^{p,q}(\alpha_i) = \begin{cases} 0, & i \neq q \\ \beta_p, & i = q \end{cases} = \delta_{iq}\beta_p, \quad 1 \leq p \leq m, 1 \leq q \leq n$$

$$\begin{matrix} E^{1,1}, & \ldots, & E^{1,n} \\ \vdots & \ddots & \vdots \\ E^{m,1}, & \ldots, & E^{m,n} \end{matrix}$$

- Spans: T:V➔W.
  - We show

$$T\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p$$

$$T = U = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}$$

$$U(\ _j) = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{p,q}E^{p,q}(\ _j)$$

$$= \sum_{p=1}^{m}\left(\sum_{q=1}^{n} A_{p,q}\ _{j,q}\right)(\ _p)$$

$$= \sum_{p=1}^{m} A_{pj}\ _p = T\ _j, j = 1, .., m$$

$$T = U$$

- Independence
  - Suppose

$$
\begin{aligned}
U &= \textstyle\sum_p \sum_q A_{pq} E^{p,q} = 0 \\
U\alpha_j &= 0 \\
\textstyle\sum_p A_{pj}\beta_p &= 0 \\
\{\beta_p\} & \quad \text{independent} \\
A_{pj} &= 0 \text{ for all } p, j
\end{aligned}
$$

- Example: $V=F^m$ $W=F$
  - $M_{mxn}(F)$ is isomorphic to $L(F^m,F^n)$ as vector spaces. Both dimensions equal mn.
  - $E^{p,q}$ is the mxn matrix with 1 at (p,q) and 0 everywhere else.
  - Any matrix is a linear combination of $E^{p,q}$.

- Theorem. T:V➔W, U:W➔Z.
  UT:V➔Z defined by UT(a)= U(T(a)) is linear.
- Definition: Linear operator T:V➔V.
- L(V,V) has a multiplication.
  - Define $T^0=I$, $T^n=T\ldots T$. n times.
  - Example: A mxn matrix B pxm matrix
    T defined by T(X)=AX. U defined by U(Y)=BY.
    Then UT(X) = BAX. Thus
    UT is defined by BA if T is defined by A and U by B.
  - Matrix multiplication is defined to mimic composition.