Rank

- "row-rank of a matrix" counts the max. number of linearly independent rows.
- "column-rank of a matrix" counts the max. number of linearly independent columns.
- One application: Given a large system of linear equations, count the number of essentially different equations.
  - The number of essentially different equations is just the row-rank of the augmented matrix.

Evaluating the row-rank by definition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$\Rightarrow$ Row-Rank = 2

Calculation of row-rank via RREF

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{\text{Row reductions}} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-rank = 2

Row-rank = 2

Because row reductions
do not affect the number
of linearly independent rows

Calculation of column-rank by definition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

List all combinations of columns

$\Rightarrow$ Column-Rank = 2

Linearly independent??

$$\begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 2 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 6 & 8 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 6 & 8 \end{bmatrix} \quad \text{N}$$

Theorem

Given any matrix, its row-rank and column-rank are equal.

In view of this property, we can just say the “rank of a matrix”. It means either the row-rank of column-rank.

Why row-rank = column-rank?

- If some column vectors are linearly dependent, they remain linearly dependent after any elementary row operation
- For example, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{bmatrix}$ are linearly dependent

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 10 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 10 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Why row-rank = column-rank?

- Any row operation does not change the column- rank.
- By the same argument, apply to the transpose of the matrix, we conclude that any column operation does not change the row-rank as well.

Why row-rank = column-rank?

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply row reductions.
row-rank and column-rank
do not change.

$$\downarrow$$

Apply column reductions.
row-rank and column-rank
do not change.

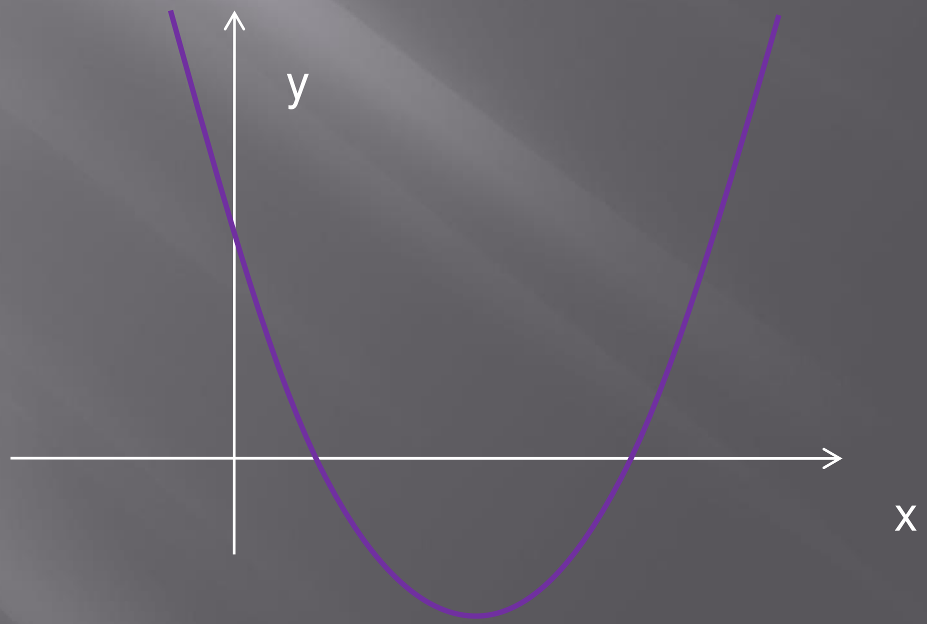
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The top-left corner is
an identity matrix.

The row-rank and column-rank of this
“normal form” is certainly
the size of this identity submatrix,
and are therefore equal.

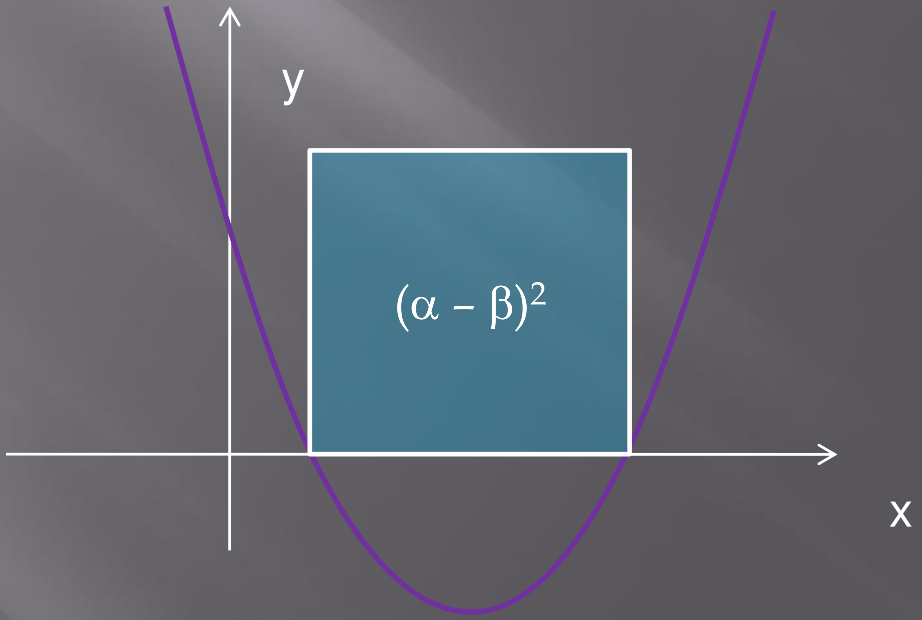
Discriminant of a quadratic equation

- $y = ax^2+bx+c$
- Discriminant of $ax^2+bx+c = b^2-4ac$.
- It determines whether the roots are distinct or not



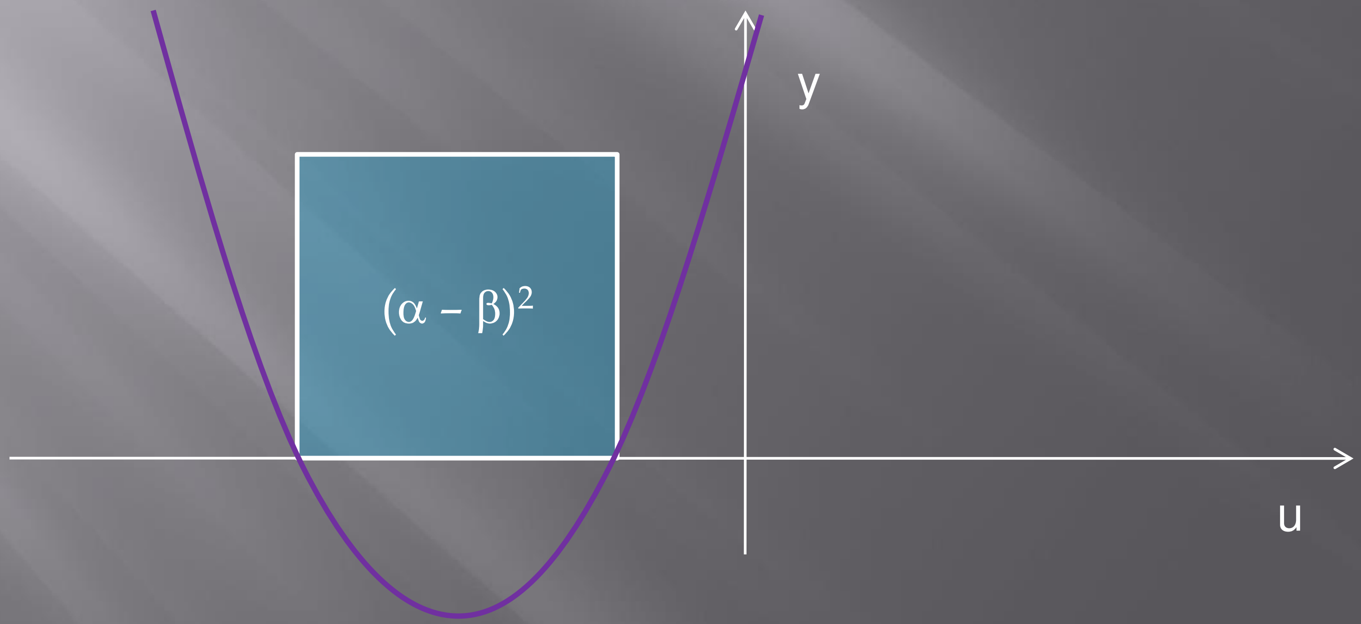
Discriminant measures the separation of roots

- $y = x^2+bx+c$. Let the roots be α and β.
- $y = (x - \alpha)(x - \beta)$. Discriminant $= (\alpha - \beta)^2$.
- Discriminant is zero means that the two roots coincide.



Discriminant is invariant under translation

- If we substitute u= x – t into y = ax^2+bx+c, (t is any real constant), then the discriminant of a$(u+t)^2$+b(u+t)+c, as a polynomial in u, is the same as before.



Determinant of a square matrix

- The determinant of a square matrix determine whether the matrix is invertible or not.
  - Zero determinant: not invertible
  - Non-zero determinant: invertible.

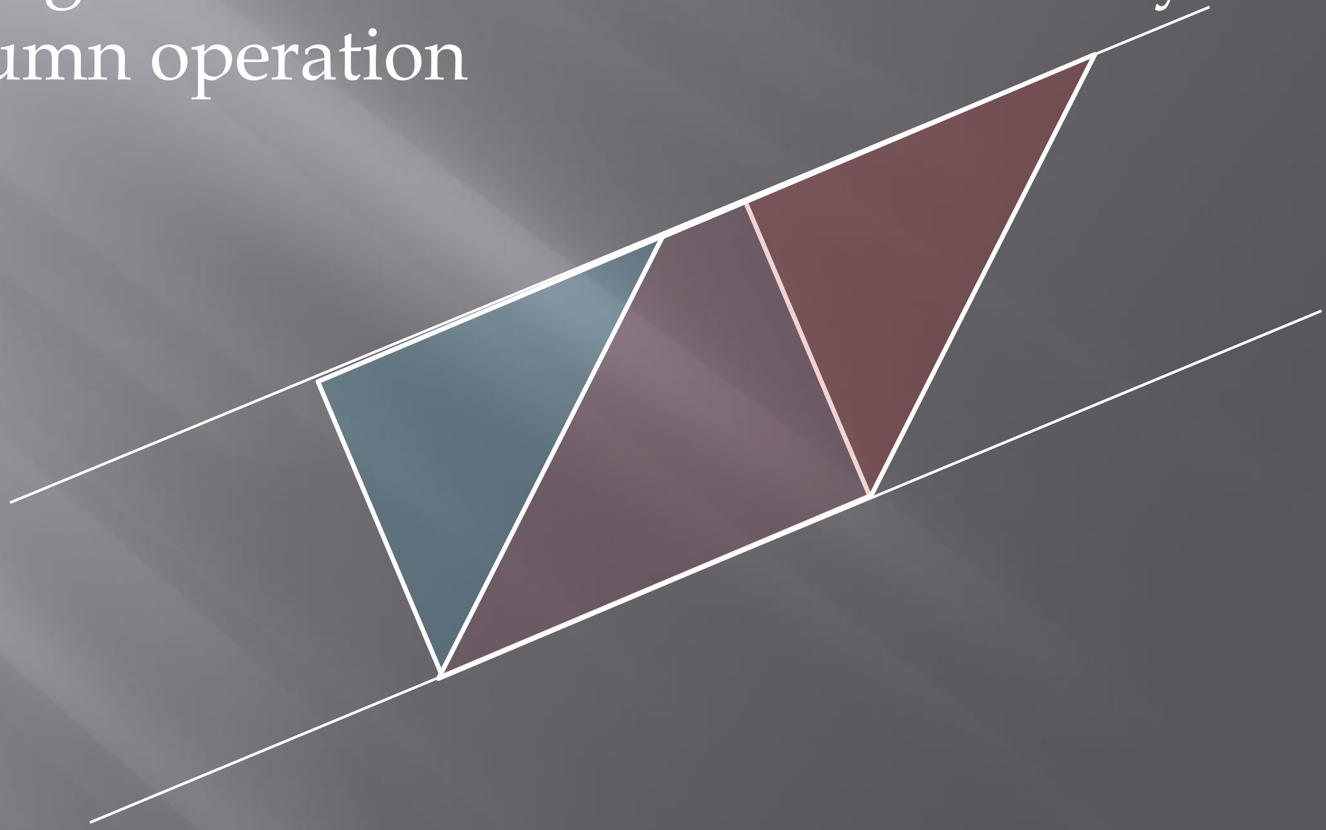
$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Determinant measure the area

- 2×2 determinant measures the area of a parallelogram.
- 3×3 determinant measures the volume of a parallelopiped.
- n×n determinant measures the "volume" of some "parallelogram" in n-dimension.
- Determinant is zero means that the columns vectors lie in some lower-dimensional space.

Determinant is invariant under shearing action

- Shearing action = third kind of elementary row or column operation



Rank of a rectangular matrix

- The rank of a matrix counts the maximal number of linearly independent rows.
- It also counts the maximal number of linearly independent columns.
- It is an integer.
- If the matrix is $m \times n$, then the rank is an integer between 0 and $\min(m,n)$.

Rank is invariant under row and column operations

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Rank = 2

$$\begin{bmatrix} 10 & 20 \\ 3 & 4 \end{bmatrix}$$

Rank = 2

$$\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

Rank = 2

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Rank = 2

$$\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$$

Rank = 2

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

Rank = 2

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

Rank = 2

Comparison between det and rank

DETERMINANT

- Real number
- Defined to square matrix only
- Non-zero det implies existence of inverse.
- When det is zero, we only know that all the columns (or rows) together are linearly dependent, but don't know any information about subset of columns (or rows) which are linearly independent.

RANK

- Integer
- Defined to any rectangular matrix
- When applied to $n \times n$ square matrix, rank=n implies existence of inverse.

Basis: Definition

- For *any* given vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$

if there is one and only one choice for the coefficients $c_1, c_2, \ldots, c_k$, such that

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + c_3 \begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{n3} \end{bmatrix} + \ldots + c_k \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix}$$

we say that these k vectors form a basis of $\mathbb{R}^n$.

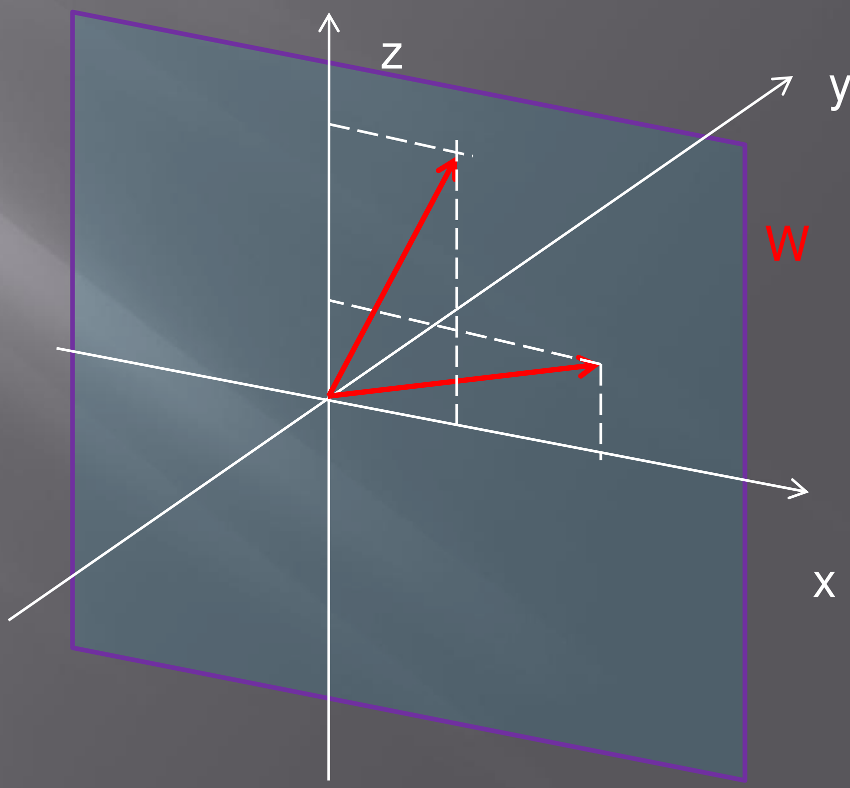
Yet another interpretation of rank

- Recall that a subspace W in $\mathbb{R}^n$ is a subset which is
  - Closed under addition: Sum of any two vectors in W stay in W.
  - Closed under scalar multiplication: scalar multiple of any vector in W stays in W as well.

Basis and dimension

- A basis of a subspace W is a set of linearly independent vectors which span W.
- A rigorous definition of the dimension is:

Dim(W) = the number of vectors in a basis of W.



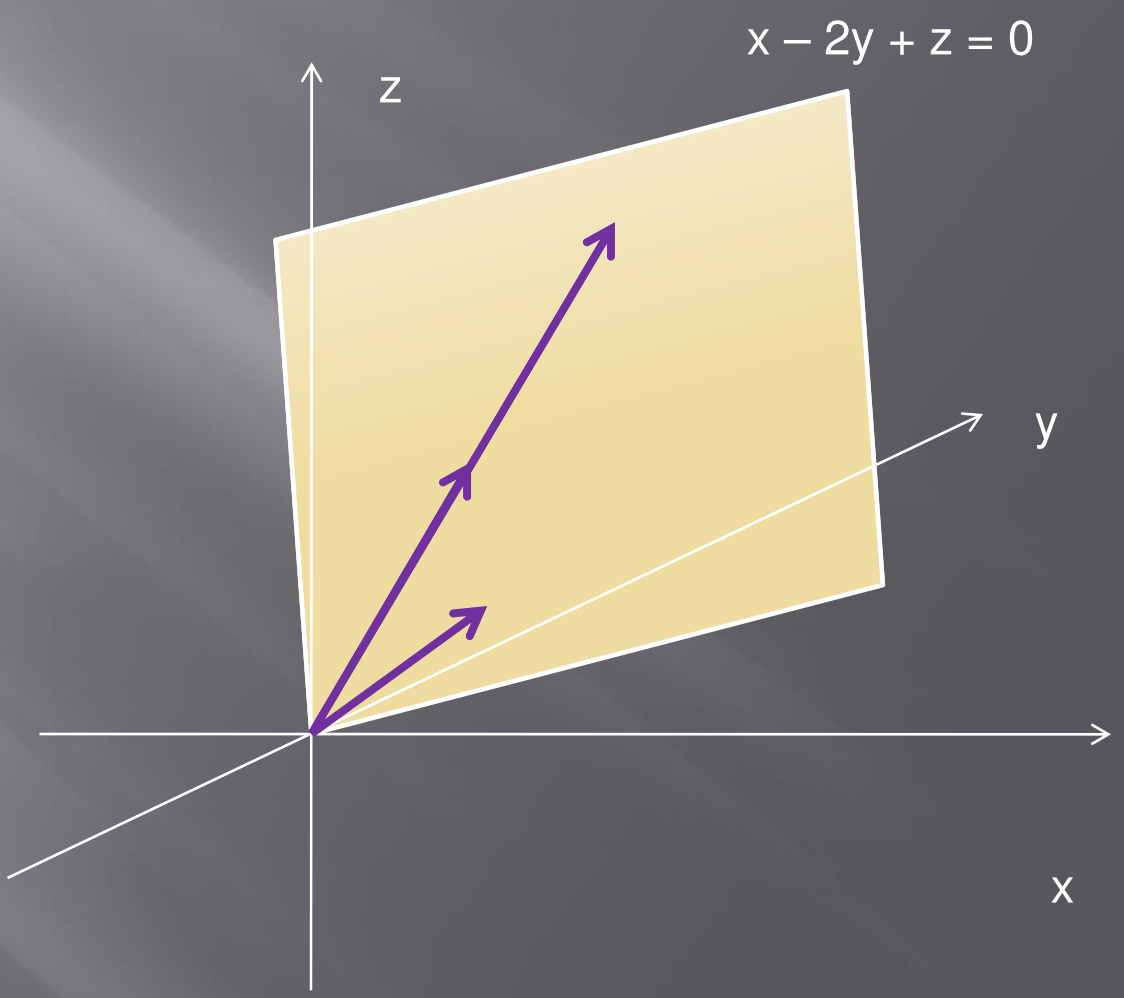
Rank as dimension

- In this context, the *rank of a matrix* is the dimension of the subspace spanned by the rows of this matrix.
  - The least number of row vectors required to span the subspace spanned by the rows.
- The rank is also the dimension of the subspace spanned by the column of this matrix.
  - The least number of column vectors required to span the subspace spanned by the columns

Example

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Rank = 2



The three row vectors lie on the same plane.

Two of them is enough to describe the plane.