

Special functions

Gamma and Beta Functions

The factorial function

$$\int_0^{\infty} e^{-ax} dx = -\frac{1}{\alpha} e^{-\alpha x} \Big|_0^{\infty} = \frac{1}{\alpha} \quad (\alpha > 0)$$

$$\int_0^{\infty} xe^{-ax} dx = -\frac{1}{\alpha} xe^{-\alpha x} \Big|_0^{\infty} - \int_0^{\infty} \left(-\frac{1}{\alpha} e^{-\alpha x} \right) dx = \frac{1}{\alpha^2}.$$

Similarly, $\int_0^{\infty} x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3}, \quad \int_0^{\infty} x^3 e^{-\alpha x} dx = \frac{2 \cdot 3}{\alpha^4}$

$$\int_0^{\infty} x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}} \rightarrow \int_0^{\infty} x^n e^{-x} dx = n! \quad (\alpha = 1)$$

Definition of the gamma function: recursion relation

– Gamma function $\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx, \quad p > 0.$

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx = (n-1)!,$$

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx = n!.$$

– Recursion relation $\Gamma(p+1) = \int_0^{\infty} x^p e^{-x} dx = p!, \quad p > -1.$

$$\Gamma(p+1) = p\Gamma(p)$$

– Example $\Gamma(9/4) = (5/4)\Gamma(5/4) = (5/4)(1/4)\Gamma(1/4)$

$$\text{so, } \Gamma(1/4) \div \Gamma(9/4) = 16/5.$$

The Gamma function of negative numbers

$$\Gamma(p) = \frac{1}{p} \Gamma(p+1) \quad (p < 0)$$

– Example

$$\Gamma(-0.3) = \frac{1}{-0.3} \Gamma(0.7), \quad \Gamma(-1.3) = \frac{1}{(-0.3)(-1.3)} \Gamma(0.7).$$

cf. $\Gamma(p) = \frac{1}{p} \Gamma(p+1) \rightarrow \infty$ as $p \rightarrow 0$.

– Using the above relation,

- 1) $\Gamma(p = \text{negative integers}) \rightarrow \text{infinite}$.
- 2) For $p < 0$, the sign changes alternatively in the intervals between negative integers

Some important formulas involving gamma functions

$$- \Gamma(1/2) = \sqrt{\pi}$$

$$\text{(prove)} \quad \Gamma(1/2) = \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-t} dt = \int_0^{\infty} \frac{1}{y} e^{-y^2} 2y dy = 2 \int_0^{\infty} e^{-y^2} dy.$$

$$[\Gamma(1/2)]^2 = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r dr d\theta = \pi.$$

$$- \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}.$$

Beta functions

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx, \quad p > 0, \quad q > 0. \quad \text{cf.} \quad B(p, q) = B(q, p)$$

$$i) \quad B(p, q) = \int_0^a \left(\frac{y}{a}\right)^{p-1} \left(1 - \frac{y}{a}\right)^{q-1} \frac{dy}{a} = \frac{1}{a^{p+q-1}} \int_0^a y^{p-1} (a-y)^{q-1} dy. \quad (x = y/a)$$

$$ii) \quad B(p, q) = 2 \int_0^{\pi/2} (\sin \theta)^{2p-1} (\cos \theta)^{2q-1} d\theta. \quad (x = \sin^2 \theta)$$

$$iii) \quad B(p, q) = \int_0^\infty \frac{y^{p-1} dy}{(1+y)^{p+q}}. \quad (x = y/(1+y))$$

Beta functions in terms of gamma functions

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Prove)

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt = 2 \int_0^\infty y^{2p-1} e^{-y^2} dy, \quad \Gamma(q) = 2 \int_0^\infty x^{2q-1} e^{-x^2} dx$$

$$\Gamma(p)\Gamma(q) = 4 \int_0^\infty \int_0^\infty x^{2q-1} y^{2p-1} e^{-(x^2+y^2)} dx dy$$

$$= 4 \int_0^\infty \int_0^{\pi/2} (r \cos \theta)^{2q-1} (r \sin \theta)^{2p-1} e^{-r^2} r dr d\theta$$

$$= 4 \int_0^\infty r^{2p+2q-1} e^{-r^2} dr \int_0^{\pi/2} (\cos \theta)^{2q-1} (\sin \theta)^{2p-1} d\theta = \frac{1}{2} \Gamma(p+q) \cdot \frac{1}{2} B(p, q).$$

- Example $I = \int_0^{\infty} \frac{x^3 dx}{(1+x)^5}$ cf. $B(p, q) = \int_0^{\infty} \frac{y^{p-1} dy}{(1+y)^{p+q}}$.

$$p + q = 5, \quad p - 1 = 3 \rightarrow p = 4, \quad q = 1.$$

$$\frac{\Gamma(4)\Gamma(1)}{\Gamma(5)} = \frac{3!}{4!} = \frac{1}{4}.$$

LEGENDRE'S DUPLICATION FORMULA

$$\Gamma(1+z)\Gamma\left(z+\frac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

General proof in §13.3.

Proof for $z = n = 1, 2, 3, \dots$:

(Case $z = 0$ is proved by inspection.)

$$\Gamma\left(n+\frac{1}{2}\right) = \left[\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left[\frac{3}{2}\frac{1}{2}\right] \right] \sqrt{\pi}$$

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

$$= \frac{1}{2^n} \left[(2n-1)(2n-3)\cdots\left[\frac{3}{2}\frac{1}{2}\right] \right] \sqrt{\pi} = \frac{1}{2^n} (2n-1)!! \sqrt{\pi}$$

$$\Gamma(1+n) = n! = \frac{(2n)!!}{2^n} \quad \rightarrow \quad \Gamma(1+n)\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!!}{2^{2n}} (2n-1)!! \sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)}{2^{2n}} \sqrt{\pi}$$

13.3. THE BETA FUNCTION

Beta Function :

$$B(p, q) \equiv \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)} = B(q, p)$$

$$\Gamma(p) \Gamma(q) = 4 \int_0^{\infty} ds e^{-s^2} s^{2p-1} \int_0^{\infty} dt e^{-t^2} t^{2q-1}$$

$$\Gamma(z) = 2 \int_0^{\infty} ds e^{-s^2} s^{2z-1}$$

$$\begin{aligned} s &= r \cos \theta \\ t &= r \sin \theta \end{aligned} \quad \rightarrow \quad ds dt = \begin{vmatrix} \partial_r s & \partial_\theta s \\ \partial_r t & \partial_\theta t \end{vmatrix} dr d\theta = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} dr d\theta = r dr d\theta$$

$$\Gamma(p) \Gamma(q) = 4 \int_0^{\infty} r dr \int_0^{\pi/2} d\theta e^{-r^2} r^{2(p+q-1)} \cos^{2p-1} \theta \sin^{2q-1} \theta$$

$$= 2 \Gamma(p+q) \int_0^{\pi/2} d\theta \cos^{2p-1} \theta \sin^{2q-1} \theta$$

$$p = m + 1$$

$$q = n + 1$$

$$\rightarrow B(p, q) = 2 \int_0^{\pi/2} d\theta \cos^{2p-1} \theta \sin^{2q-1} \theta$$

$$\frac{m! n!}{(m+n+1)!} = 2 \int_0^{\pi/2} d\theta \cos^{2m+1} \theta \sin^{2n+1} \theta$$

ALTERNATE FORMS : DEFINITE INTEGRALS

$$B(p+1, q+1) = 2 \int_0^{\pi/2} d\theta \cos^{2p+1} \theta \sin^{2q+1} \theta$$

$$t = \cos^2 \theta \quad \rightarrow \quad B(p+1, q+1) = \int_0^1 d \cos^2 \theta \cos^{2p} \theta \sin^{2q} \theta = \int_0^1 dt t^p (1-t)^q$$

$$t = x^2 \quad \rightarrow \quad B(p+1, q+1) = 2 \int_0^1 dx x^{2p+1} (1-x^2)^q$$

$$t = \frac{u}{1+u} \quad \rightarrow \quad dt = \left(\frac{1}{1+u} - \frac{u}{(1+u)^2} \right) du = \frac{1}{(1+u)^2} du \quad 1-t = \frac{1}{1+u}$$

$$B(p+1, q+1) = \int_0^{1/2} du \frac{1}{(1+u)^2} \left(\frac{u}{1+u} \right)^p \left(\frac{1}{1+u} \right)^q = \int_0^{1/2} du \frac{u^p}{(1+u)^{p+q+2}}$$

To be used in integral rep. of Bessel (Ex.14.1.17)
& hypergeometric (Ex.18.5.12) functions

DERIVATION: LEGENDRE DUPLICATION FORMULA

$$B(p+1, q+1) = \int_0^1 dt \, t^p (1-t)^q = 2 \int_0^1 dx \, x^{2p+1} (1-x^2)^q$$

$$t = \frac{1}{2}(1+s)$$

$$\begin{aligned} \rightarrow B\left(z + \frac{1}{2}, z + \frac{1}{2}\right) &= \int_0^1 dt \, t^{z-1/2} (1-t)^{z-1/2} = \left(\frac{1}{2}\right)^{2z} \int_{-1}^1 ds \, (1+s)^{z-1/2} (1-s)^{z-1/2} \\ &= \left(\frac{1}{2}\right)^{2z} \int_{-1}^1 ds \, (1-s^2)^{z-1/2} = \left(\frac{1}{2}\right)^{2z} 2 \int_0^1 ds \, (1-s^2)^{z-1/2} = \left(\frac{1}{2}\right)^{2z} B\left(\frac{1}{2}, z + \frac{1}{2}\right) \end{aligned}$$

$$\rightarrow \frac{\Gamma\left(z + \frac{1}{2}\right) \Gamma\left(z + \frac{1}{2}\right)}{\Gamma(2z+1)} = \left(\frac{1}{2}\right)^{2z} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(z + \frac{1}{2}\right)}{\Gamma(z+1)}$$

$$\Gamma(z+1) \Gamma\left(z + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z}} \Gamma(2z+1)$$