Dirichlet’s Integral and Applications

DIRICHLET'S INTEGRAL:

If l, m, n are all positive, then the triple integral

$$\iiint_V x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(\mathrm{l})\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(l+m+n+1)}$$

Where V is the region $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq 1$.

Note:

$$\iiint_V x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} h^{l+m+n}$$

Where V is the domain, $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq h$

Corollary:

Dirichlet's theorem for n variables, the theorem status that

$$\iiint \dots \int x_1^{l_1-1}\, x_2^{l_2-1} \dots x_n^{l_n-1}\, dx_1\, dx_2\, dx_3 \dots dx_n = \frac{\Gamma l_1 \Gamma l_2 \Gamma l_3 \dots \Gamma l_n}{\Gamma(1 + l_1 + l_2 + \cdots + l_n)}\, h^{l_1+l_2+\cdots+l_n}$$

Liouville's extension of dirichlet theorem:

$$\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$$

$$=\frac{\Gamma(\mathrm{l})\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(\mathrm{l}+\mathrm{m}+\mathrm{n})}\int_{h_1}^{h_2} f(u)\,u^{l+m+n-1}du$$

Example1: Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$,
the integral being taken throughout the volume bounded by $x = 0, y = 0, z = 0, x + y + z = 1$.

Solution: By Liouville's theorem, when $0 < x + y + z < 1$

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iiint \frac{x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz}{(x+y+z+1)^3} \qquad (0 \le x + y + z \le 1)$$

$$= \frac{\Gamma 1 \Gamma 1 \Gamma 1}{\Gamma(l+m+n)} \int_0^1 \frac{1}{(u+1)^3} u^{3-1}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{u^2}{(u+1)^3}\,du$$

$$= \int_0^1 \left[\frac{1}{u+1} - \frac{2}{(u+1)^2} + \frac{1}{(u+1)^3}\right] du \qquad \text{(Partial fractions)}$$

$$= \frac{1}{2}\left[\log(u+1) + \frac{2}{u+1} - \frac{1}{2(u+1)^2}\right]_0^1$$

$$= \frac{1}{2}\left[\log 2 + 2\left(\frac{1}{2} - 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$

$$\therefore \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$$

Example 2: Find the mass of an octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the density at any point being $\rho = k\, x\, y\, z$.

Solution: $\quad \text{Mass} = \iiint \rho \; dv = \iiint (k\,x\,y\,z) dx\, dy\, dz$

$$= k \iiint (x\, dx)(y\, dx)(z\, dz) \qquad \_\_\_\_\_\_ (1)$$

Putting $\quad \frac{x^2}{a^2} = u, \frac{y^2}{b^2} = v, \frac{z^2}{c^2} = w$ and $u + v + w = 1$

So that $\quad \frac{2x\, dx}{a^2} = du, \frac{2y\, dy}{b^2} = dv, \frac{2z\, dz}{c^2} = dw$

$$\text{Mass} = k \iiint \left(\frac{a^2\, du}{2}\right)\left(\frac{b^2\, dv}{2}\right)\left(\frac{c^2\, dw}{2}\right)$$

$$= \frac{k\, a^2 b^2 c^2}{8} \iiint du\, dv\, dw, \qquad \text{Where } u + v + w \le 1$$

$$= \frac{k\, a^2 b^2 c^2}{8} \iiint u^{l-1}\, v^{l-1}\, w^{l-1} du\, dv\, dw$$

$$= \frac{k\, a^2 b^2 c^2}{8} \frac{\Gamma 1 \Gamma 1 \Gamma 1}{\Gamma 3+1} = \frac{k\, a^2 b^2 c^2}{8 \times 6}$$

$$= \frac{k\, a^2 b^2 c^2}{48}$$

$$\therefore \; \boldsymbol{Mass} = \frac{\boldsymbol{k\, a^2 b^2 c^2}}{\boldsymbol{48}}$$