Curl of a Vector

If $\underset{\sim}{A} = a_x \underset{\sim}{i} + a_y \underset{\sim}{j} + a_z \underset{\sim}{k}$, the curl of $\underset{\sim}{A}$ is defined by

$$curl\ \underset{\sim}{A} = \nabla \times \underset{\sim}{A}$$

$$= \left(\frac{\partial}{\partial x} \underset{\sim}{i} + \frac{\partial}{\partial y} \underset{\sim}{j} + \frac{\partial}{\partial z} \underset{\sim}{k} \right) \times \left(a_x \underset{\sim}{i} + a_y \underset{\sim}{j} + a_z \underset{\sim}{k} \right)$$

$$\Rightarrow curl\ \underset{\sim}{A} = \nabla \times \underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}.$$

Example :

If $\underset{\sim}{A} = (y^4 - x^2 z^2)\underset{\sim}{i} + (x^2 + y^2)\underset{\sim}{j} - x^2 yz\,\underset{\sim}{k}$,

determine $curl\ \underset{\sim}{A}$ at (1,3,–2).

Solution

$$curl\, \underset{\sim}{A} = \nabla \times \underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^4 - x^2 z^2 & x^2 + y^2 & -x^2 yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-x^2 yz) - \frac{\partial}{\partial z}(x^2 + y^2) \right) \underset{\sim}{i}$$

$$- \left(\frac{\partial}{\partial x}(-x^2 yz) - \frac{\partial}{\partial z}(y^4 - x^2 z^2) \right) \underset{\sim}{j}$$

$$+ \left(\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2 z^2) \right) \underset{\sim}{k}$$

$$= -x^2 z\, \underset{\sim}{i} - (-2xyz + 2x^2 z)\, \underset{\sim}{j} + (2x - 4y^3)\, \underset{\sim}{k}\, .$$

At (1,3,-2),

$$\underset{\sim}{curl\ A} = -(1)^2(-2)\underset{\sim}{i} - (-2(1)(3)(-2) + 2(1)^2(-2))\underset{\sim}{j}$$

$$+ (2(1) - 4(3)^3)\underset{\sim}{k}$$

$$= 2\underset{\sim}{i} - 8\underset{\sim}{j} - 106\underset{\sim}{k}.$$

Exercise:

If $\underset{\sim}{A} = (xy^3 - y^2z^2)\underset{\sim}{i} + (x^2 + z^2)\underset{\sim}{j} - x^2yz^2\underset{\sim}{k}$,

determine $curl\ \underset{\sim}{A}$ at point (1,2,3).

Answer

$$curl\ \underset{\sim}{A} = (-x^2z^2 - 2z)\,\underset{\sim}{i} - (-2xyz^2 + 2y^2z)\,\underset{\sim}{j}$$

$$+ (2x - 3xy^2 + 2yz^2)\,\underset{\sim}{k}.$$

$$\text{At } (1,2,3),\quad curl\ \underset{\sim}{A} = -15\,\underset{\sim}{i} + 12\,\underset{\sim}{j} + 26\,\underset{\sim}{k}.$$

Remark

$\underset{\sim}{A}$ is a vector function and

$curl\ \underset{\sim}{A}$ is also a vector function.