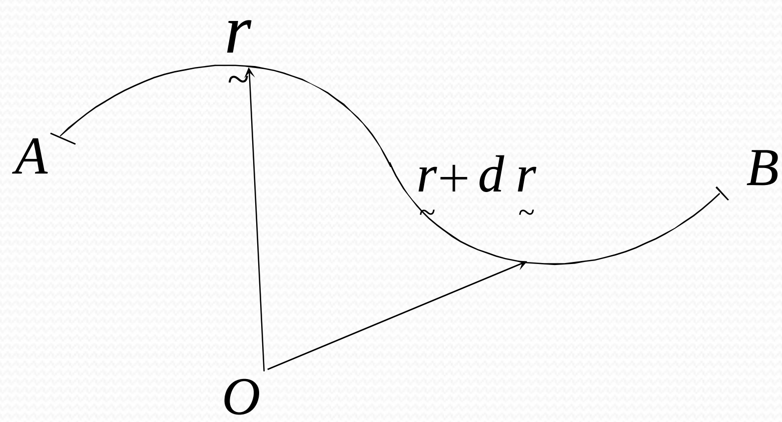
Line Integral

Ordinary integral $\int f(x)\,dx$, we integrate along the x-axis. But for line integral, the integration is along a curve.

$$\int f(s)\,ds = \int f(x, y, z)\,ds$$



Scalar Field, V Integral

If there exists a scalar field V along a curve C, then the line integral of V along C is defined by

$$\int_c V \, d\underset{\sim}{r}$$

where $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}$.

Example:

If $V = xy^2 z$ and a curve C is given by

$$x = 3u, \quad y = 2u^2, \quad z = u^3,$$

then find $\int_c V \, d \underset{\sim}{r}$ along C

from $\mathrm{A} = (0,0,0)$ to $\mathrm{B} = (3,2,1)$.

Solution

Given $V = xy^2 z$

$$= (3u)(2u^2)^2(u^3) = 12u^8.$$

And, $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}$

$$= 3\,du\,\underset{\sim}{i} + 4u\,du\,\underset{\sim}{j} + 3u^2 du\,\underset{\sim}{k}.$$

At A $= (0,0,0)$, $3u = 0,\ 2u^2 = 0,\ u^3 = 0,$

$\Rightarrow u = 0.$

At B $= (3,2,1)$, $3u = 3,\ 2u^2 = 2,\ u^3 = 1,$

$\Rightarrow u = 1.$

$$\therefore \int_A^B V \, d\underset{\sim}{r} = \int_{u=0}^{u=1} (12u^8)(3du\,\underset{\sim}{i} + 4udu\,\underset{\sim}{j} + 3u^2du\,\underset{\sim}{k})$$

$$= \int_0^1 36u^8du\,\underset{\sim}{i} + \int_0^1 48u^9du\,\underset{\sim}{j} + \int_0^1 36u^{10}du\,\underset{\sim}{k}$$

$$= \left[4u^9\right]_0^1 \underset{\sim}{i} + \left[\frac{24}{5}u^{10}\right]_0^1 \underset{\sim}{j} + \left[\frac{36}{11}u^{11}\right]_0^1 \underset{\sim}{k}$$

$$= 4\underset{\sim}{i} + \frac{24}{5}\underset{\sim}{j} + \frac{36}{11}\underset{\sim}{k}.$$

Exercise:

If $V = x^2yz^2$ and the curve C is given by

$$x = 4u, \quad y = 3u^3, \quad z = 2u^2,$$

calculate $\int_c V\, d\, \underset{\sim}{r}$ along the curve C

from A = (0,0,0) to B = (4,3,2).

Answer

$$\int_A^B V\, d\, \underset{\sim}{r} = \frac{384}{5}\underset{\sim}{i} + 144\,\underset{\sim}{j} + \frac{768}{11}\underset{\sim}{k}.$$

Vector Field Integral

Let a vector field

$$\underset{\sim}{F} = F_x \underset{\sim}{i} + F_y \underset{\sim}{j} + F_z \underset{\sim}{k}$$

and

$$d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}.$$

The scalar product is written as

$$\underset{\sim}{F}.d\underset{\sim}{r}$$

$$\underset{\sim}{F}.d\underset{\sim}{r} = (F_x \underset{\sim}{i} + F_y \underset{\sim}{j} + F_z \underset{\sim}{k}).(dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k})$$

$$= F_x dx + F_y dy + F_z dz.$$

If a vector field $\underset{\sim}{F}$ is along the curve C,

then the line integral of $\underset{\sim}{F}$ along the curve C

from a point A to another point B is given by

$$\int_c \underset{\sim}{F} \,.\, d\,\underset{\sim}{r} = \int_c F_x\, dx + \int_c F_y\, dy + \int_c F_z\, dz.$$

Example :

Calculate $\int_c \underset{\sim}{F} \, . \, d\underset{\sim}{r}$ from $\mathrm{A} = (0,0,0)$ to $\mathrm{B} = (4,2,1)$

along the curve $x = 4t, y = 2t^2, z = t^3$ if

$\underset{\sim}{F} = x^2 \mathrm{y}\, \underset{\sim}{i} + x\mathrm{z}\, \underset{\sim}{j} - 2\mathrm{yz}\, \underset{\sim}{k}.$

Solution

$$\text{Given } \underset{\sim}{F} = x^2 y\, \underset{\sim}{i} + xz\, \underset{\sim}{j} - 2yz\, \underset{\sim}{k}$$

$$= (4t)^2 (2t^2)\, \underset{\sim}{i} + (4t)(t^3)\, \underset{\sim}{j} - 2(2t^2)(t^3)\, \underset{\sim}{k}$$

$$= 32t^4\, \underset{\sim}{i} + 4t^4\, \underset{\sim}{j} - 4t^5\, \underset{\sim}{k}.$$

$$\text{And } d\,\underset{\sim}{r} = dx\, \underset{\sim}{i} + dy\, \underset{\sim}{j} + dz\, \underset{\sim}{k}$$

$$= 4\,dt\, \underset{\sim}{i} + 4t\,dt\, \underset{\sim}{j} + 3t^2 dt\, \underset{\sim}{k}.$$

Then

$$\underset{\sim}{F}.d\underset{\sim}{r} = (32t^4\,\underset{\sim}{i} + 4t^4\,\underset{\sim}{j} - 4t^5\,\underset{\sim}{k})(4\,dt\,\underset{\sim}{i} + 4t\,dt\,\underset{\sim}{j} + 3t^2 dt\,\underset{\sim}{k})$$

$$= (32t^4)(4dt) + (4t^4)(4tdt) + (-4t^5)(3t^2 dt)$$

$$= 128t^4 dt + 16t^5 dt - 12t^7 dt$$

$$= (128t^4 + 16t^5 - 12t^7)\,dt.$$

At A $= (0,0,0)$, $4t = 0,\ 2t^2 = 0,\ t^3 = 0,$

$$\Rightarrow t = 0.$$

and, at B $= (4,2,1)$, $4t = 4,\ 2t^2 = 2,\ t^3 = 1,$

$$\Rightarrow t = 1.$$

$$\therefore \int_A^B \underset{\sim}{F} \,.\, d\,\underset{\sim}{r} = \int_{t=0}^{t=1} (128t^4 + 16t^5 - 12t^7)\,dt$$

$$= \left[\frac{128}{5} t^5 + \frac{8}{3} t^6 - \frac{3}{2} t^8 \right]_0^1$$

$$= \frac{128}{5} + \frac{8}{3} - \frac{3}{2}$$

$$= 26\frac{23}{30}.$$

Exercise:

If $\underset{\sim}{F} = xy^2 \underset{\sim}{i} - yz \underset{\sim}{j} + 3x^2 z \underset{\sim}{k}$,

calculate $\int_c \underset{\sim}{F} \cdot d\underset{\sim}{r}$

from A = (0,0,0) to B = (1,2,3) on the

curve $x = t, y = 2t^2, z = 3t^3$.

Answer $\int_A^B \underset{\sim}{F} \cdot d\underset{\sim}{r} = 7\frac{61}{168}$.

* Double Integral *

Example

Given $f(x,y) = 4 - y^2$ in region R bounded by a straight line $x = 0$, $y = x$ and $y = 2$.

Find $\iint\limits_R f(x,y)\,dA$ in both order integrals.

Answer $\iint\limits_R f(x,y)\,dA = 4 \text{ unit}^2.$

Example

Using double integral, find the area of a region bounded by $y = 5 - x^2$ and $y = x + 3$.

Answer The area of the region $= 4\frac{1}{2}$ unit^2.