Surface Integral

Scalar Field, V Integral

If scalar field V exists on surface S, surface integral V of S is defined by

$$\int_S V d\underset{\sim}{S} = \int_S V \underset{\sim}{n}\, dS$$

where

$$\underset{\sim}{n} = \frac{\nabla S}{\left|\nabla S\right|}$$

Example :

Scalar field $V = x\,y\,z$ defeated on the surface $S : x^2 + y^2 = 4$ between $z = 0$ and $z = 3$ in the first octant.

Evaluate $\int_S V d\,\underset{\sim}{S}$

Solution

Given $S : x^2 + y^2 = 4$, so grad S is

$$\nabla S = \frac{\partial S}{\partial x}\underset{\sim}{i} + \frac{\partial S}{\partial y}\underset{\sim}{j} + \frac{\partial S}{\partial z}\underset{\sim}{k} = 2x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j}$$

Also,

$$\left|\nabla S\right| = \sqrt{(2x)^2 + (2y)^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

Therefore,

$$\underset{\sim}{n} = \frac{\nabla S}{\left|\nabla S\right|} = \frac{2x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j}}{4} = \frac{1}{2}(x\,\underset{\sim}{i} + y\,\underset{\sim}{j})$$

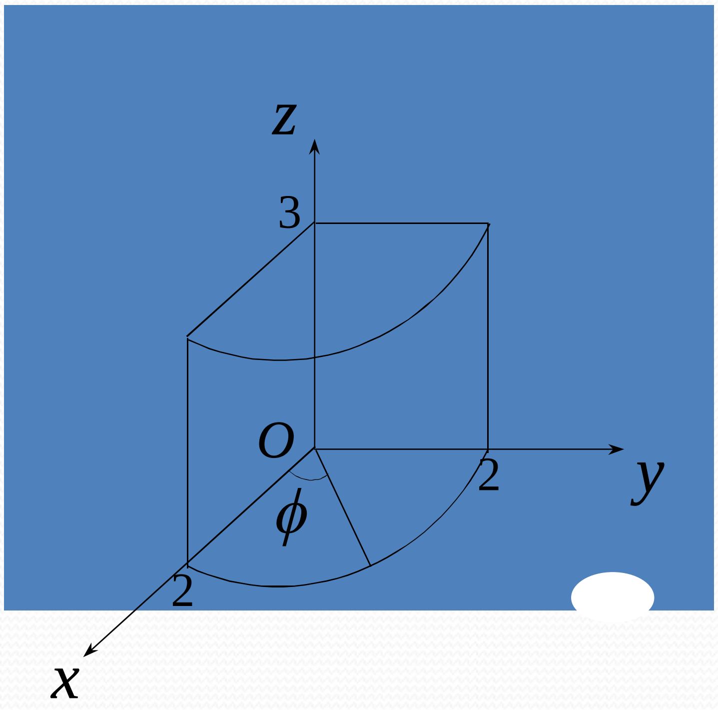
Then,

$$\int_S V\,\underset{\sim}{n}\,dS = \int_S xyz\left(\frac{1}{2}\right)(x\,\underset{\sim}{i} + y\,\underset{\sim}{j})dS$$

$$= \frac{1}{2}\int (x^2yz\,\underset{\sim}{i} + xy^2z\,\underset{\sim}{j})dS$$

Surface $S : x^2 + y^2 = 4$ is bounded by $z = 0$ and $z = 3$ that is a cylinder with z-axis as a cylinder axes and radius, $\rho = \sqrt{4} = 2.$

So, we will use polar coordinate of cylinder to find the surface integral.



Polar Coordinate for Cylinder

$$x = \rho\cos\phi = 2\cos\phi$$

$$y = \rho\sin\phi = 2\sin\phi$$

$$z = z$$

$$dS = \rho\, d\phi\, dz$$

where $0 \le \phi \le \frac{\pi}{2}$ (1st octant) and $0 \le z \le 3$

Using polar coordinate of cylinder,

$$x^2yz = (2\cos\phi)^2(2\sin\phi)z = 8z\cos^2\phi\sin\phi$$

$$xy^2z = (2\cos\phi)(2\sin\phi)^2(z) = 8z\sin^2\phi\cos\phi$$

From

$$\int_S V\,\underset{\sim}{n}\,dS = \frac{1}{2}\int_S (x^2yz\,\underset{\sim}{i} + xy^2z\,\underset{\sim}{j})dS = \int_S V d\underset{\sim}{S}$$

Therefore,

$$\int_S V\, d\underset{\sim}{S} = \frac{1}{2}\int_{\phi=0}^{\frac{\pi}{2}}\int_{z=0}^{3}(8z\cos^2\phi\sin\phi\,\underset{\sim}{i} + 8z\sin^2\phi\cos\phi\,\underset{\sim}{j})(2)\,dz\,d\phi$$

$$= 8\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}z^2\cos^2\phi\sin\phi\,\underset{\sim}{i} + \frac{1}{2}z^2\sin^2\phi\cos\phi\,\underset{\sim}{j}\right]_0^3 d\phi$$

$$= 8\int_0^{\frac{\pi}{2}}\left[\frac{9}{2}\cos^2\phi\sin\phi\,\underset{\sim}{i} + \frac{9}{2}\sin^2\phi\cos\phi\,\underset{\sim}{j}\right] d\phi$$

$$= 8\times\frac{9}{2}\int_0^{\frac{\pi}{2}}\left[\cos^2\phi\sin\phi\,\underset{\sim}{i} + \sin^2\phi\cos\phi\,\underset{\sim}{j}\right] d\phi$$

$$= 36\left[\frac{\cos^3\phi\sin\phi}{3(-\sin\phi)}\underset{\sim}{i} + \frac{\sin^3\phi\cos\phi}{3(\cos\phi)}\underset{\sim}{j}\right]_0^{\frac{\pi}{2}}$$

$$= 12(\underset{\sim}{i} + \underset{\sim}{j})$$

Exercise:

$If\ V\ is\ a\ scalar\ field\ where\ V = xyz^2,\ evaluate$

$\int_S V\ d\underset{\sim}{S}\ for\ surface\ S\ that\ region\ bounded\ by\ x^2 + y^2 = 9$

$between\ z = 0\ and\ z = 2\ in\ the\ first\ octant.$

$Answer: 24(\underset{\sim}{i} + \underset{\sim}{j})$

Vector Field, Integral

If vector field defeated on surface S, surface integral $\underset{\sim}{F}$ of S is defined as

$$\int_S \underset{\sim}{F} \,.\, d\underset{\sim}{S} = \int_S \underset{\sim}{F} \,.\, \underset{\sim}{n}\, dS$$

where $\underset{\sim}{n} = \dfrac{\nabla S}{|\nabla S|}$

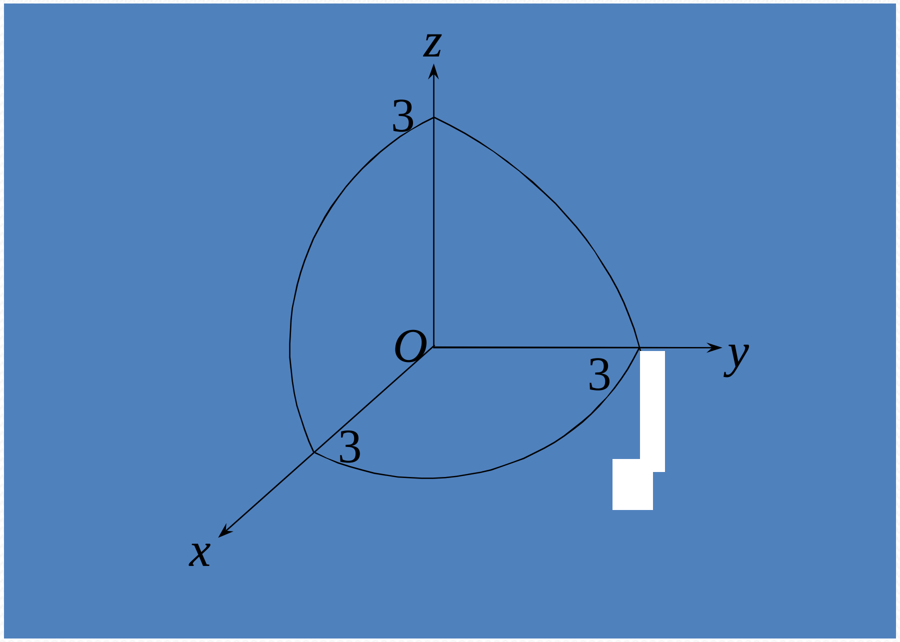
Example:

Vector field $\underset{\sim}{F} = y\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ defeated on surface $S : x^2 + y^2 + z^2 = 9$ and bounded by $x = 0, y = 0, z = 0$ in the first octant.

Evaluate $\int_S \underset{\sim}{F} . d\underset{\sim}{S}$.

Solution

Given $S: x^2 + y^2 + z^2 = 9$ is bounded by $x = 0$, $y = 0$, $z = 0$ in the 1st octant. This refer to sphere with center at (0,0,0) and radius, $r = 3$, in the 1st octant.



So, grad S is

$$\nabla S = \frac{\partial S}{\partial x}\underset{\sim}{i} + \frac{\partial S}{\partial y}\underset{\sim}{j} + \frac{\partial S}{\partial z}\underset{\sim}{k}$$

$$= 2x\underset{\sim}{i} + 2y\underset{\sim}{j} + 2z\underset{\sim}{k},$$

and

$$|\nabla S| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$$

$$= 2\sqrt{x^2 + y^2 + z^2}$$

$$= 2\sqrt{9} = 6.$$

$$\therefore \qquad \underset{\sim}{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\underset{\sim}{i} + 2y\underset{\sim}{j} + 2z\underset{\sim}{k}}{6}$$

$$= \frac{1}{3}(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}).$$

Therefore,

$$\int_S \underset{\sim}{F} . d\underset{\sim}{S} = \int_S \underset{\sim}{F} . \underset{\sim}{n}\, dS$$

$$= \int_S (y\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})\left(\frac{1}{3}\right)(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})\, dS$$

$$= \frac{1}{3}\int_S (xy + 2y + z)\, dS.$$

Using polar coordinate of sphere,

$$x = r\sin\theta\cos\phi = 3\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi = 3\sin\theta\sin\phi$$

$$z = r\cos\theta = 3\cos\theta$$

$$dS = r^2\sin\theta\, d\theta\, d\phi = 9\sin\theta\, d\theta\, d\phi$$

$$\text{where} \quad 0 \le \theta,\ \phi \le \frac{\pi}{2}.$$

$$\therefore \int_S \underset{\sim}{F} . d\underset{\sim}{S} = \frac{1}{3}\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}} [(3\sin\theta\cos\phi)(3\sin\theta\sin\phi)$$

$$+2(3\sin\theta\sin\phi)+3\cos\theta][9\sin\theta]d\theta\, d\phi$$

$$= 9\int_{\phi=0}^{\frac{\pi}{2}}\int_{\theta=0}^{\frac{\pi}{2}} [3\sin^3\theta\sin\phi\cos\phi$$

$$+2\sin^2\theta\sin\phi+\sin\theta\cos\theta]d\theta d\phi$$

$$\vdots$$

$$\vdots$$

$$= 9\left(1+\frac{3\pi}{4}\right)$$

Exercise:

Evaluate $\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$ on S, where $\underset{\sim}{F} = x\underset{\sim}{i} + 2z\underset{\sim}{j} + y\underset{\sim}{k}$

and S is a surface of the region bounded by

$x^2 + y^2 + z^2 = 4$, $x = 0$, $y = 0$ and $z = 0$ in the 1st octant.

$$Answer : 8\left(\frac{\pi}{6} + 1\right)$$