Elementary Vector Analysis

Definition 2.1 (***Scalar*** and ***vector***)

Scalar is a quantity that has magnitude but not direction.

For instance *mass, volume, distance*

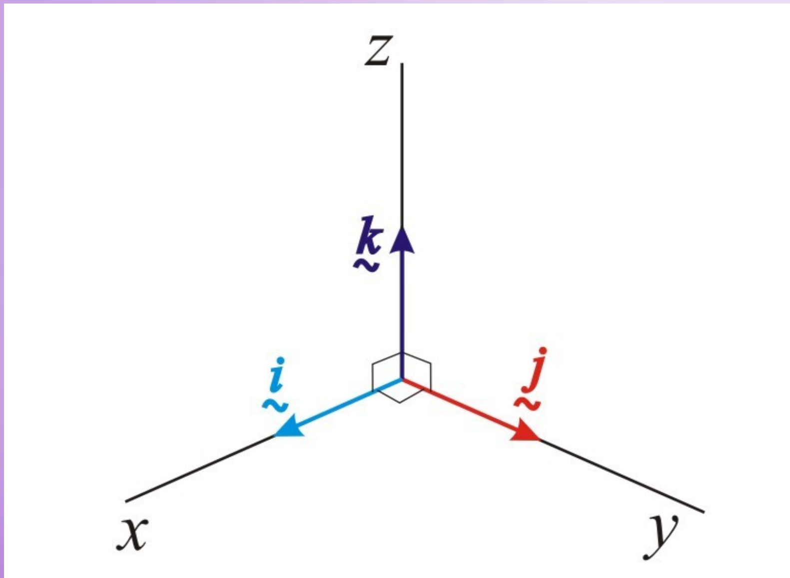
Vector is a directed quantity, one with both magnitude and direction.

For instance *acceleration, velocity, force*

We represent a vector as an arrow from the origin $\boldsymbol{O}$ to a point $\boldsymbol{A}$.

The length of the arrow is the magnitude of the vector written as $|\overrightarrow{OA}|$ or $|\underset{\sim}{a}|$.

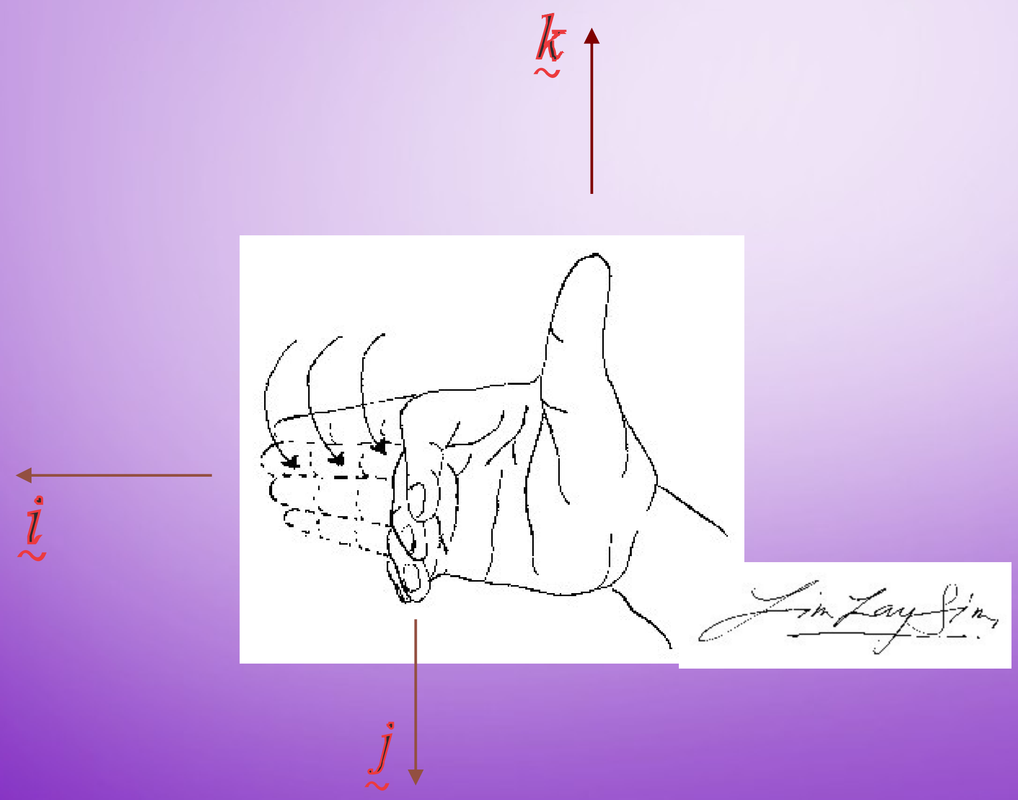
Basic Vector System



Unit vectors $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$

- Perpendicular to each other
- In the positive directions of the axes
- have magnitude (length) 1

Define a *basic vector system* and form a *right-handed set*, i.e



Magnitude of vectors

Let P = (x, y, z). Vector $\overrightarrow{OP} = \underset{\sim}{p}$ is defined by

$$\overrightarrow{OP} = \underset{\sim}{p} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j} + z\,\underset{\sim}{k}$$

$$= [x, y, z]$$

with magnitude (length)

$$|\overrightarrow{OP}| = |\underset{\sim}{p}| = \sqrt{x^2 + y^2 + z^2}$$

Calculation of Vectors

1. Vector Equation

Two vectors are equal if and only if the corresponding components are equals

Let $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$.

Then

$$\underline{a} = \underline{b} \Leftrightarrow a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3$$

2. Addition and Subtraction of Vectors

$$\underline{a} \pm \underline{b} = (a_1 \pm b_1)\underline{i} + (a_2 \pm b_2)\underline{j} + (a_3 \pm b_3)\underline{k}$$

3. Multiplication of Vectors by Scalars

If α is a scalar, then

$$\alpha \underline{b} = (\alpha b_1)\underline{i} + (\alpha b_2)\underline{j} + (\alpha b_3)\underline{k}$$

Example:

Given $\underset{\sim}{p} = 5\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{q} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$. Find

a) $\underset{\sim}{p} + \underset{\sim}{q}$

b) $\underset{\sim}{p} - \underset{\sim}{q}$

c) Magnitude of vector $\underset{\sim}{p}$

d) $2\underset{\sim}{q} - 10\underset{\sim}{p}$

Vector Products

If $\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$ and $\underset{\sim}{b} = b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k}$,

1) Scalar Product (Dot product)

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

or $\underset{\sim}{a} . \underset{\sim}{b} = |a||b| \cos\theta$, θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

2) Vector Product (Cross product)

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \left(a_2 b_3 - a_3 b_2\right) \underset{\sim}{i} - \left(a_1 b_3 - a_3 b_1\right) \underset{\sim}{j} + \left(a_1 b_2 - a_2 b_1\right) \underset{\sim}{k}$$

Vector Differential Calculus

- Let *A* be a vector depending on parameter u,

$$\underset{\sim}{A}(u) = a_x(u)\,\underset{\sim}{i} + a_y(u)\,\underset{\sim}{j} + a_z(u)\,\underset{\sim}{k}$$

- The derivative of *A*(*u*) is obtained by differentiating each component separately,

$$\frac{d\,\underset{\sim}{A}}{du} = \frac{da_x}{du}\underset{\sim}{i} + \frac{da_y}{du}\underset{\sim}{j} + \frac{da_z}{du}\underset{\sim}{k}$$

- The nth derivative of vector $\underset{\sim}{A}$ is given by

$$\frac{d^n \underset{\sim}{A}}{du^n} = \frac{d^n a_x}{du^n}\underset{\sim}{i} + \frac{d^n a_y}{du^n}\underset{\sim}{j} + \frac{d^n a_z}{du^n}\underset{\sim}{k}.$$

- The magnitude of $\frac{d^n \underset{\sim}{A}}{du^n}$ is

$$\left|\frac{d^n \underset{\sim}{A}}{du^n}\right| = \sqrt{\left(\frac{d^n a_x}{du^n}\right)^2 + \left(\frac{d^n a_y}{du^n}\right)^2 + \left(\frac{d^n a_z}{du^n}\right)^2}$$

Example :

The position of a moving particle at time t is given by $x = 4t + 3,\ y = t^2 + 3t,\ z = t^3 + 5t^2$. Obtain

- The velocity and acceleration of the particle.
- The magnitude of both velocity and acceleration at $t = 1$.

Solution

- The parameter is *t*, and the position vector is

$$\underset{\sim}{r}(t) = (4t+3)\,\underset{\sim}{i} + (t^2+3t)\,\underset{\sim}{j} + (t^3+5t^2)\,\underset{\sim}{k}.$$

- The velocity is given by

$$\frac{d\,\underset{\sim}{r}}{dt} = 4\,\underset{\sim}{i} + (2t+3)\,\underset{\sim}{j} + (3t^2+10t)\,\underset{\sim}{k}.$$

- The acceleration is

$$\frac{d^2\,\underset{\sim}{r}}{dt^2} = 2\,\underset{\sim}{j} + (6t+10)\,\underset{\sim}{k}.$$

- At $t = 1$, the velocity of the particle is

$$\frac{d\,\underset{\sim}{r}(1)}{dt} = 4\,\underset{\sim}{i} + (2(1)+3)\,\underset{\sim}{j} + (3(1)^2 + 10(1))\,\underset{\sim}{k}$$

$$= 4\,\underset{\sim}{i} + 5\,\underset{\sim}{j} + 13\,\underset{\sim}{k}.$$

and the magnitude of the velocity is

$$\left|\frac{d\,\underset{\sim}{r}(1)}{dt}\right| = \sqrt{4^2 + 5^2 + 13^2}$$

$$= \sqrt{210}.$$

- At $t = 1$, the acceleration of the particle is

$$\frac{d^2 \underset{\sim}{r}(1)}{dt^2} = 2\underset{\sim}{j} + (6(1)+10)\underset{\sim}{k}$$

$$= 2\underset{\sim}{j} + 16\underset{\sim}{k}.$$

and the magnitude of the acceleration is

$$\left|\frac{d^2 \underset{\sim}{r}(1)}{dt^2}\right| = \sqrt{2^2 + 16^2}$$

$$= 2\sqrt{65}.$$

Differentiation of Two Vectors

If both $\underset{\sim}{A}(u)$ and $\underset{\sim}{B}(u)$ are vectors, then

$$a)\quad \frac{d}{du}(c\,\underset{\sim}{A}) = c\,\frac{d\,\underset{\sim}{A}}{du}$$

$$b)\quad \frac{d}{du}(\underset{\sim}{A}+\underset{\sim}{B}) = \frac{d\,\underset{\sim}{A}}{du}+\frac{d\,\underset{\sim}{B}}{du}$$

$$c)\quad \frac{d}{du}(\underset{\sim}{A}.\underset{\sim}{B}) = \underset{\sim}{A}.\frac{d\,\underset{\sim}{B}}{du}+\frac{d\,\underset{\sim}{A}}{du}.\underset{\sim}{B}$$

$$d)\quad \frac{d}{du}(\underset{\sim}{A}\times\underset{\sim}{B}) = \underset{\sim}{A}\times\frac{d\,\underset{\sim}{B}}{du}+\frac{d\,\underset{\sim}{A}}{du}\times\underset{\sim}{B}$$

Partial Derivatives of a Vector

- If vector $\underset{\sim}{A}$ depends on more than one parameter, i.e

$$\underset{\sim}{A}(u_1, u_2, \cdots, u_n) = a_x(u_1, u_2, \cdots, u_n)\underset{\sim}{i} + a_y(u_1, u_2, \cdots, u_n)\underset{\sim}{j} + a_z(u_1, u_2, \cdots, u_n)\underset{\sim}{k}$$

- Partial derivative of $\underset{\sim}{A}$ with respect to u_1 is given by

$$\frac{\partial \underset{\sim}{A}}{\partial u_1} = \frac{\partial a_x}{\partial u_1}\underset{\sim}{i} + \frac{\partial a_y}{\partial u_1}\underset{\sim}{j} + \frac{\partial a_z}{\partial u_1}\underset{\sim}{k},$$

$$\frac{\partial^2 \underset{\sim}{A}}{\partial u_1 \partial u_2} = \frac{\partial^2 a_x}{\partial u_1 \partial u_2}\underset{\sim}{i} + \frac{\partial^2 a_y}{\partial u_1 \partial u_2}\underset{\sim}{j} + \frac{\partial^2 a_z}{\partial u_1 \partial u_2}\underset{\sim}{k}$$

e.t.c.

Example

If $\underset{\sim}{F} = 3uv^2 \underset{\sim}{i} + (2u^2 - v)\underset{\sim}{j} + (u^3 + v^2)\underset{\sim}{k}$

then

$$\frac{\partial \underset{\sim}{F}}{\partial u} = 3v^2 \underset{\sim}{i} + 4u \underset{\sim}{j} + 3u^2 \underset{\sim}{k},$$

$$\frac{\partial \underset{\sim}{F}}{\partial v} = 6uv \underset{\sim}{i} - \underset{\sim}{j} + 2v \underset{\sim}{k}, \qquad \frac{\partial^2 \underset{\sim}{F}}{\partial u^2} = 4\underset{\sim}{j} + 6u \underset{\sim}{k},$$

$$\frac{\partial^2 \underset{\sim}{F}}{\partial v^2} = 6u \underset{\sim}{i} + 2\underset{\sim}{k}, \qquad \frac{\partial^2 \underset{\sim}{F}}{\partial u \partial v} = \frac{\partial^2 \underset{\sim}{F}}{\partial v \partial u} = 6v \underset{\sim}{i}$$

Exercise:

If $\underset{\sim}{F} = 2u^2v\,\underset{\sim}{i} + (3u - v^3)\,\underset{\sim}{j} + (u^3 + 3v^2)\,\underset{\sim}{k}$

then

$$\frac{\partial \underset{\sim}{F}}{\partial u} = \cdots, \qquad \frac{\partial \underset{\sim}{F}}{\partial v} = \cdots$$

$$\frac{\partial^2 \underset{\sim}{F}}{\partial u^2} = \cdots, \qquad \frac{\partial^2 \underset{\sim}{F}}{\partial v^2} = \cdots$$

$$\frac{\partial^2 \underset{\sim}{F}}{\partial u \partial v} = \cdots, \qquad \frac{\partial^2 \underset{\sim}{F}}{\partial v \partial u} = \cdots$$

Vector Integral Calculus

- The concept of vector integral is the same as the integral of real-valued functions except that the result of vector integral is a vector.

If $\underset{\sim}{A}(u) = a_x(u)\,\underset{\sim}{i} + a_y(u)\,\underset{\sim}{j} + a_z(u)\,\underset{\sim}{k}$

then

$$\int_a^b \underset{\sim}{A}(u)\,du = \int_a^b a_x(u)\,du\,\underset{\sim}{i} + \int_a^b a_y(u)\,du\,\underset{\sim}{j} + \int_a^b a_z(u)\,du\,\underset{\sim}{k}.$$

Example :

If $\underset{\sim}{F} = (3t^2 + 4t)\,\underset{\sim}{i} + (2t - 5)\,\underset{\sim}{j} + 4t^3\,\underset{\sim}{k}$,

calculate $\int_1^3 \underset{\sim}{F}\,dt$.

Answer

$$\int_1^3 \underset{\sim}{F}\,dt = \int_1^3 (3t^2 + 4t)\,dt\,\underset{\sim}{i} + \int_1^3 (2t - 5)\,dt\,\underset{\sim}{j} + \int_1^3 4t^3\,dt\,\underset{\sim}{k}$$

$$= [t^3 + 2t^2]_1^3\,\underset{\sim}{i} + [t^2 - 5t]_1^3\,\underset{\sim}{j} + [t^4]_1^3\,\underset{\sim}{k}$$

$$= 42\,\underset{\sim}{i} - 2\,\underset{\sim}{j} + 80\,\underset{\sim}{k}.$$

Exercise:

If $\underset{\sim}{F} = (t^3 + 3t)\,\underset{\sim}{i} + 2t^2\,\underset{\sim}{j} + (t-4)\,\underset{\sim}{k}$,

calculate $\int_0^1 \underset{\sim}{F}\,dt$.

Answer

$$\int_0^1 \underset{\sim}{F}\,dt = \int_0^1 (t^3 + 3t)\,dt\,\underset{\sim}{i} + \int_0^1 2t^2\,dt\,\underset{\sim}{j} + \int_0^1 (t-4)\,dt\,\underset{\sim}{k}$$

$$= \ldots$$

$$= \ldots$$

$$= \frac{7}{4}\underset{\sim}{i} + \frac{2}{3}\underset{\sim}{j} - \frac{7}{2}\underset{\sim}{k}.$$

Del Operator Or Nabla (Symbol ∇)

- Operator ∇ is called vector differential operator, defined as

$$\nabla = \left(\frac{\partial}{\partial x} \underset{\sim}{i} + \frac{\partial}{\partial y} \underset{\sim}{j} + \frac{\partial}{\partial z} \underset{\sim}{k} \right).$$