Volume Integral

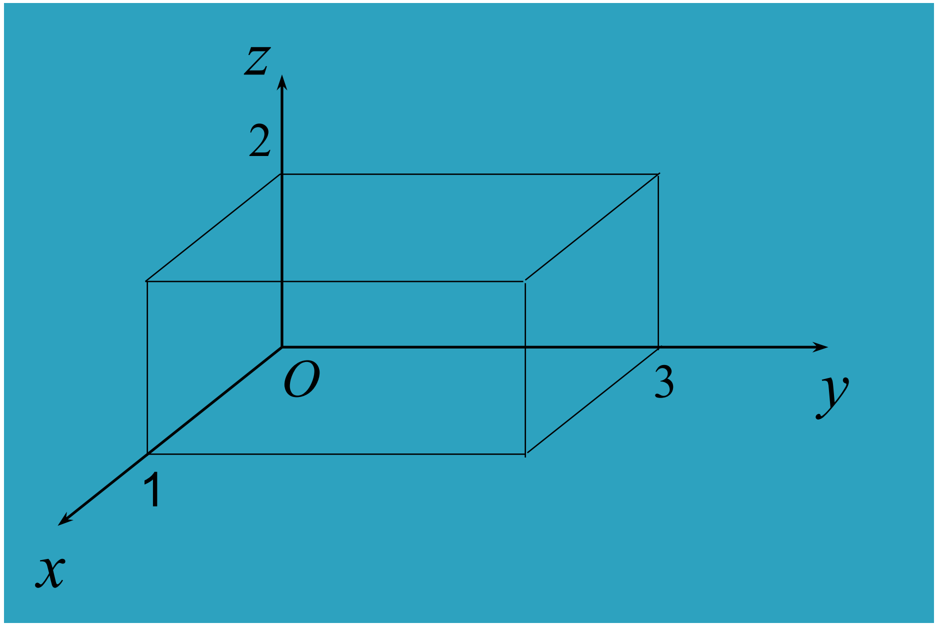
Scalar Field, *F* Integral

If V is a closed region and F is a scalar field in region V, volume integral F of V is

$$\int_V F\,dV = \iiint_V F\,dx\,dy\,dz$$

Example :

Scalar function $F = 2\,x$ defeated in one cubic that has been built by planes $x = 0$, $x = 1$, $y = 0$, $y = 3$, $z = 0$ and $z = 2$. Evaluate volume integral F of the cubic.



Solution

$$\int_V F dV = \int_{z=0}^{2}\int_{y=0}^{3}\int_{x=0}^{1} 2x dx dy dz$$

$$= 2\int_{z=0}^{2}\int_{y=0}^{3}\left[\frac{x^2}{2}\right]_0^1 dy dz$$

$$= 2\int_{z=0}^{2}\int_{y=0}^{3}\frac{1}{2} dy dz$$

$$= 2.\frac{1}{2}\int_{z=0}^{2}[y]_0^3 dz$$

$$= \int_{z=0}^{2} 3dz = 3[z]_0^2 = 6$$

Volume Integral

If V is a closed region and $\underset{\sim}{F}$, vector field in region V, Volume integral $\underset{\sim}{F}$ of V is

$$\int_V \underset{\sim}{F}\, dV = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \underset{\sim}{F}\, dz\,dy\,dx$$

Example:

Evaluate $\int_V \underset{\sim}{F}\, dV$, where V is a region bounded by $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 2$, and also given $\underset{\sim}{F} = 2z\,\underset{\sim}{i} + y\,\underset{\sim}{k}$

Solution

If $x = y = 0$, plane $2x + y + z = 2$ intersects z-axis at $z = 2$.

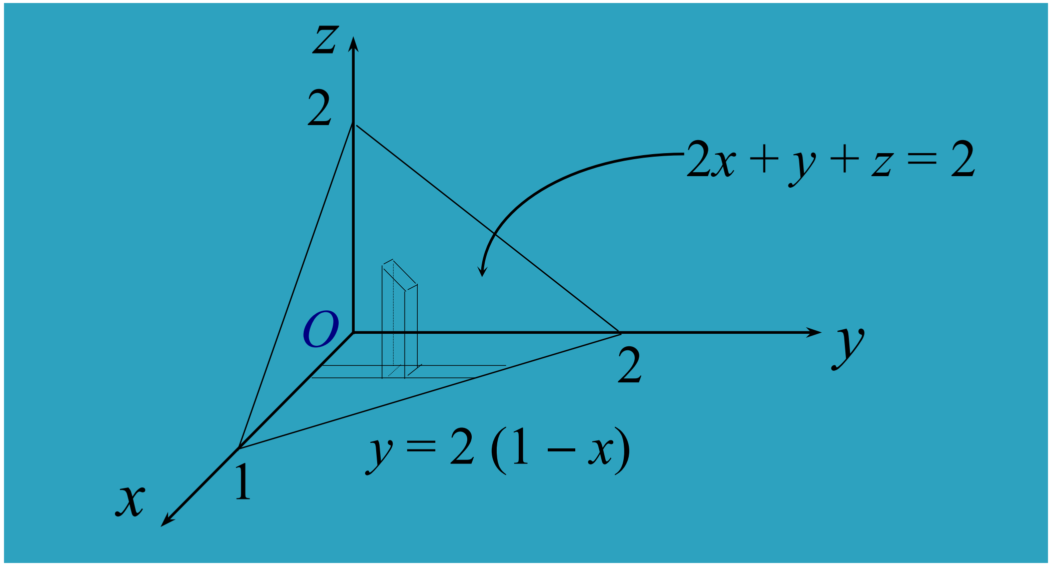
$\Rightarrow$ (0,0,2)

If $x = z = 0$, plane $2x + y + z = 2$ intersects y-axis at $y = 2$.

$\Rightarrow$ (0,2,0)

If $y = z = 0$, plane $2x + y + z = 2$ intersects x-axis at $x = 1$.

$\Rightarrow$ (1,0,0)



We can generate this integral in 3 steps :

1. Line Integral from $x = 0$ to $x = 1$.

2. Surface Integral from line $y = 0$ to line $y = 2(1-x)$.

3. Volume Integral from surface $z = 0$ to surface $2x + y + z = 2$ that is $z = 2\,(1-x) - y$

Therefore,

$$\int_V \underset{\sim}{F}\, dV = \int_{x=0}^{1}\int_{y=0}^{2(1-x)}\int_{z=0}^{2(1-x)-y} \underset{\sim}{F}\, dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{2(1-x)}\int_{z=0}^{2(1-x)-y} (2z\,\underset{\sim}{i} + y\,\underset{\sim}{k})\, dz\,dy\,dx$$

$$\vdots$$

$$\vdots$$

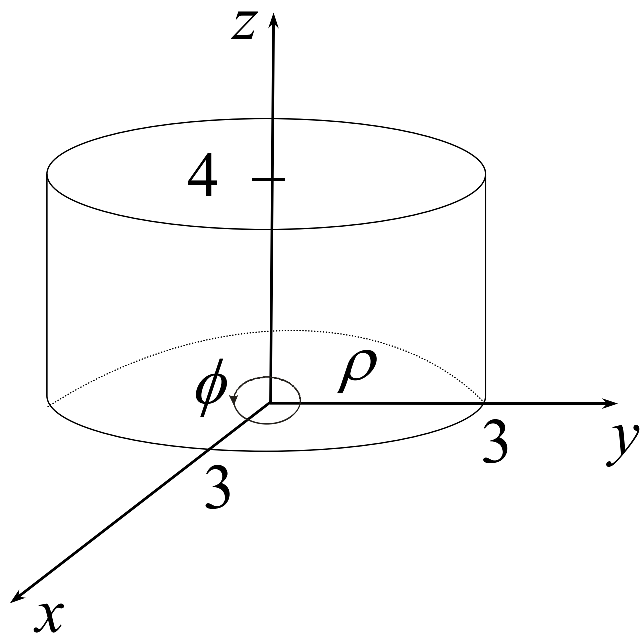
$$= \frac{2}{3}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{k}$$

Example:

Evaluate $\int_V \underset{\sim}{F}\, dV$ where $\underset{\sim}{F} = 2\underset{\sim}{i} + 2z\,\underset{\sim}{j} + y\,\underset{\sim}{k}$

and V is region bounded by $z = 0$, $z = 4$ and

$x^2 + y^2 = 9$



Using polar coordinate of cylinder,

$$x = \rho\cos\phi; \; y = \rho\sin\phi; \; z = z;$$

$$dV = \rho\, d\rho\, d\phi\, dz$$

where

$$0 \le \rho \le 3, \; 0 \le \phi \le 2\pi, \; 0 \le z \le 4$$

Therefore,

$$\int_V \underset{\sim}{F}\, dV = \iiint_V (2\underset{\sim}{i} + 2z\underset{\sim}{j} + y\underset{\sim}{k})\, dx dy dz$$

$$= \int_{z=0}^{4} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{3} (2\underset{\sim}{i} + 2z\underset{\sim}{j} + \rho \sin\phi\, \underset{\sim}{k})\rho \, d\rho d\phi dz$$

$$\vdots$$

$$\vdots$$

$$= 72\pi\underset{\sim}{i} + 144\pi\underset{\sim}{j}$$

Exercise:

$$\text{Evaluate } \int_V \underset{\sim}{F}\, dV \text{ where } \underset{\sim}{F} = 3\underset{\sim}{i} + z\underset{\sim}{j} + 2y\underset{\sim}{k} \text{ and}$$

V *is region bounded by planes* $z = 0, z = 3$ *and surface* $x^2 + y^2 = 4.$

$$\text{Answer}: 18\pi\left(2\underset{\sim}{i} + \underset{\sim}{j}\right)$$