Non-Homogeneous Second Order Differential Equation

M-II

A non-homogeneous second order differential equation is of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

Step 1

We find the general solution of the homogeneous equation as before

The general solution to the equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

is known as the complementary function.

Step 2

Find a particular solution to the non homogeneous equation. This solution is called the **particular integral** and looks similar to *f(x)*

Step 3

The general solution of the non-homogeneous differential equation is the sum of the complementary function and the particular integral

How to find the particular integral

$f\left(x\right)=2x+1\therefore$ particular integral is $y=px+q$

$f\left(x\right)=x^2-1\therefore$ particular integral is $y=px^2+qx+r$

$note: f\left(x\right)=x^2+0x-1$

$f\left(x\right)=4e^{2x}\therefore$ particular integral is $y=pe^{2x}$

$f\left(x\right)=2\sin x+\cos x\therefore$ particular integral is $y=p\sin x+q\cos x$

$f\left(x\right)=3\sin 2x\therefore$ particular integral is $y=p\sin 2x+q\cos 2x$

$note: f\left(x\right)=3\sin 2x+0\cos 2x$

Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x + 1$$

Complementary Function

$$k^2 - k - 2 = 0$$

$$(k-2)(k+1) = 0$$

$$k = 2 \text{ and } k = -1$$

$$y = Ae^{2x} + Be^{-x}$$

Particular Integral

$$y = px + q$$

$$\frac{dy}{dx} = p \text{ and } \frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x + 1$$

$$(0) - (p) - 2(px + q) = 4x + 1$$

$$-p - 2px - 2q = 4x + 1$$

$$-2px - p - 2q = 4x + 1$$

equate coefficients

$$4x = -2px$$

$$\therefore -2p = 4 \text{ and } p = -2$$

$$1 = -p - 2q$$

$$-(-2) - 2q = 1$$

$$2 - 2q = 1$$

$$q = \frac{1}{2}$$

$y = px + q$ with $p = -2$ and $q = \frac{1}{2}$

$$y = -2x + \frac{1}{2}$$

The general solution of the non-homogeneous differential equation is the sum of the complementary function and the particular integral

$$y = Ae^{2x} + Be^{-x} - 2x + \frac{1}{2}$$

Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 1$$

Complementary Function

$$k^2 - 4k + 4 = 0$$

$$(k-2)(k-2) = 0$$

$k = 2$ equal roots

$$y = (Ax + B)e^{2x}$$

Particular Integral

$$y = px^2 + qx + r$$

$$\frac{dy}{dx} = 2px + q \text{ and } \frac{d^2y}{dx^2} = 2p$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 1$$

$$(2p) - 4(2px + q) + 4(px^2 + qx + r) = 2x^2 + 1$$

$$2p - 8px - 4q + 4px^2 + 4qx + 4r = 2x^2 + 1$$ equate coefficients

$$4px^2 + (4q - 8p)x + 2p - 4q + 4r = 2x^2 + 1$$

$$4p = 2 \therefore p = \frac{1}{2}$$

$$4q - 8p = 0 \qquad 2p - 4q + 4r = 1$$

$$4q - 8\left(\tfrac{1}{2}\right) = 0 \qquad 2\left(\tfrac{1}{2}\right) - 4(1) + 4r = 1$$

$$4q = 4 \therefore q = 1 \qquad 4r = 4 \therefore r = 1$$

$$y = px^2 + qx + r \quad \text{with } p = \frac{1}{2}\ , q = 1 \text{ and } r = 1$$

$$y = \frac{1}{2}x^2 + x + 1$$

The general solution of the non-homogeneous differential equation is the sum of the complementary function and the particular integral

$$y = \left(Ax + B\right)e^{2x} + \frac{1}{2}x^2 + x + 1$$

Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$$

Complementary Function

$$k^2 + 2k - 3 = 0$$

$$(k+3)(k-1) = 0$$

$$k = -3 \text{ or } k = 1$$

$$y = Ae^{-3x} + Be^{x}$$

Particular Integral

$$y = pe^{2x}$$

$$\frac{dy}{dx} = 2pe^{2x} \text{ and } \frac{d^2y}{dx^2} = 4pe^{2x}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$$

$$\left(4pe^{2x}\right) + 2\left(2pe^{2x}\right) - 3\left(pe^{2x}\right) = 10e^{2x}$$

$$4pe^{2x} + 4pe^{2x} - 3pe^{2x} = 10e^{2x}$$

$$5pe^{2x} = 10e^{2x}$$

$$p = 2$$

The complementary function $y = Ae^{-3x} + Be^{x}$

The particular intergral is $y = 2e^{2x}$

$$y = Ae^{-3x} + Be^{x} + 2e^{2x}$$

Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2}+16y=e^{-2x}$$

Complementary Function

$$k^2+16=0$$

$$k^2=-16=16i^2$$

$$k=\pm 4i$$

$$y=e^{0x}\left(A\sin 4x+B\cos 4x\right)$$

$$y=A\sin 4x+B\cos 4x$$

Particular Integral

$$y = pe^{-2x}$$

$$\frac{dy}{dx} = -2pe^{-2x} \text{ and } \frac{d^2y}{dx^2} = 4pe^{-2x}$$

$$\frac{d^2y}{dx^2} + 16y = e^{-2x}$$

$$\left(4pe^{-2x}\right) + 16\left(pe^{-2x}\right) = e^{-2x}$$

$$20pe^{-2x} = e^{-2x}$$

$$20p = 1$$

$$p = \frac{1}{20}$$

$$y = A\sin 4x + B\cos 4x$$

$$y = \frac{1}{20}e^{-2x}$$

$$y = A\sin 4x + B\cos 4x + \frac{1}{20}e^{-2x}$$

Find the general solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10\sin x$$

Complementary Function

$$k^2 - 4k + 3 = 0$$

$$(k-3)(k-1) = 0$$

$$k = 3 \text{ and } k = 1$$

$$y = Ae^{3x} + Be^{x}$$

Particular Integral

$y = p\sin x + q\cos x$

$\frac{dy}{dx} = p\cos x - q\sin x$ and $\frac{d^2y}{dx^2} = -p\sin x - q\cos x$

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10\sin x$

$(-p\sin x - q\cos x) - 4(p\cos x - q\sin x) + 3(p\sin x + q\cos x) = 10\sin x$

$-p\sin x - q\cos x - 4p\cos x + 4q\sin x + 3p\sin x + 3q\cos x = 10\sin x$

$-p\sin x + 4q\sin x + 3p\sin x = 10\sin x \therefore 2p + 4q = 10$

$-q\cos x - 4p\cos x + 3q\cos x = 0 \therefore -4p + 2q = 0$

$2p + 4q = 10 \quad \times 2 \qquad 4p + 8q = 20$

$-4p + 2q = 0 \qquad\qquad -4p + 2q = 0$

$10q = 20 \therefore q = 2$ and $p = 1$

$y = \sin x + 2\cos x$

$y = Ae^{3x} + Be^{x}$

$y = Ae^{3x} + Be^{x} + \sin x + 2\cos x$

Find the particular solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

$x = 0, y = 2$ and $\frac{dy}{dx} = 1$

Complementary Function

$$k^2 - 4k + 4 = 0$$

$$(k-2) = 0$$

$k = 2$ equal roots

$$y = (Ax + b)e^{2x}$$

Particular Integral

$$y = pe^x$$

$$\frac{dy}{dx} = pe^x \text{ and } \frac{d^2y}{dx^2} = pe^x$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

$$\left(pe^x\right) - 4\left(pe^x\right) + 4\left(pe^x\right) = e^x$$

$$pe^x = e^x \therefore p = 1$$

$$y = pe^x$$

$$y = \left(Ax + B\right)e^{2x}$$

$$y = e^x$$

$$y = (Ax + B)e^{2x} + e^x$$

$$y = (Ax + B)e^{2x} + e^{x} \qquad x = 0, y = 2$$

$$2 = \left(A(0) + B\right)e^{2(0)} + e^{0}$$

$$2 = B + 1 \therefore B = 1$$

$$y = (Ax + B)e^{2x} + e^{x}$$

$$\frac{dy}{dx} = Ae^{2x} + 2e^{2x}\left(Ax + B\right) + e^{x}$$

Product and Sum Rule

$$\frac{dy}{dx} = 1, x = 0 \text{ and } B = 1$$

$$1 = Ae^{2(0)} + 2e^{2(0)}\left(A(0) + 1\right) + e^{(0)}$$

$$1 = A + 2(1) + 1 \therefore A = -2$$

$$y = (1 - 2x)e^{2x} + e^{x}$$