Differential Equations

<u>Summary (Recall)</u>

- Differential Operator, which is a linear operator.
- Differential equation in linear operator form.
- Auxiliary equation in terms of differential operator form.
- Annihilator operator.
- Annihilator operators of different functions.
- Solution non-homogeneous equation with annihilator operator.

Method of variation of parameters

- This method is more general method to solve the non-homogeneous linear differential equation.
- This method can be applied to DEs where the method of undetermined coefficients fails.
- This method is not limited to the input functions that are combinations of four type of functions (constant, polynomial, exponential and trigonometric).

  Cosx, Sinx
- It is also applicable to linear DEs with variable coefficients.
- However, the particular integral (y_p) is only possible if the associated homogeneous equation can be solved.

For first order equation:

The particular solution y_p of the first order linear DE is

$$y_p = e^{-\int P dx} . \int e^{\int P dx} . f(x) dx$$

This formula can also be derived by another method, known as the variation of parameters. The basic procedure is same as discussed in the lecture on construction of a second solution.

As

$$y_1 = e^{-\int P dx}$$

is the solution of the homogeneous first order DE

$$\frac{dy}{dx} + P(x) y = 0,$$

Therefore, the general solution of the equation is

$$y = c_1 y_1(x) \, .$$

In the method of variation of parameters we assume

$$y_p = u_1(x)\, y_1(x)$$

is a particular solution of the non-homogeneous differential equation

$$\frac{dy}{dx} + P(x)\, y = f(x)$$

$y = c_1 y_1$

Notice that the parameter c_1 has been replaced by the variable $u_1(x)$. We substitute y_p in the given equation to obtain

$$u_1\left[\frac{dy_1}{dx} + P(x)y_1\right] + y_1\frac{du_1}{dx} = f(x)$$

$\frac{dy_1}{dx} + P(x)y_1 = 0$

Since,

$$\frac{dy_1}{dx} + P(x)\, y_1 = 0 \quad \text{(because } y_1 \text{ is the solution)}$$

So that we obtain

$\therefore$ $$y_1 \frac{du_1}{dx} = f(x)$$

This is a variable separable equation.

$$y_1 = e^{\int -P(x)dx}$$

Thus, $$du_1 = \frac{f(x)}{y_1(x)} dx$$

Integration gives

$$u_1(x) = \int \frac{f(x)}{y_1} dx = \int e^{\int P dx} \cdot f(x) dx \, .$$

As $$y_p = u_1(x)\, y_1(x) \, .$$

Therefore,

$$y_p = e^{-\int P dx} . \int e^{\int P dx} . f(x) dx$$

or

$$u_1 = \int \frac{f(x)}{y_1(x)} dx \, .$$

Method for Second Order Equation:

We consider the second order linear non-homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Divide by $a_2(x)$, to get equation in the standard form

$$y'' + P(x)y' + Q(x)y = f(x),$$

where

$$P(x) = \frac{a_1(x)}{a_2(x)}, Q(x) = \frac{a_0(x)}{a_2(x)}, f(x) = \frac{g(x)}{a_2(x)}$$

are continuous on some interval I.

Consider the associated homogeneous DE,

$$y'' + P(x)y' + Q(x)y = 0 \cdot$$

***Complementary function*:**

y_1, y_2

Complementary function is

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

So that

$$y_1'' + P(x) y_1' + Q(x) y_1 = 0$$

$$y_2'' + P(x) y_2' + Q(x) y_2 = 0$$

Particular Integral

For finding a particular solution y_p, we replace the parameters c_1 c_2 and in the complementary function with the unknown variables $u_1(x)$ and $u_2(x)$. So that the assumed particular integral is

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

Since we seek to determine two unknown functions u_1 and u_2, we need two equations involving these unknowns. One of these two equations results from substituting the assumed y_p in the given differential equation. We impose the other equation to simplify the first derivative and thereby the 2nd derivative of y_p

$$y_p' = u_1 y_1' + y_1 u_1' + u_2 y_2' + u_2' y_2 = \underline{u_1 y_1' + u_2 y_2'} + \underline{u_1' y_1 + u_2' y_2}$$

To avoid 2nd derivatives of u_1 and u_2, we impose the condition

$$u_1' y_1 + u_2' y_2 = 0$$

Then, $$y_p' = u_1 y_1' + u_2 y_2'$$

So that $$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

Therefore,

$$\underline{y_p'' + P\,y_p' + Q\,y_p} = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + P u_1 y_1' + P u_2 y_2' + Q u_1 y_1 + Q u_2 y_2$$

Substituting in the given non-homogeneous differential equation yields

$$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + P u_1 y_1' + P u_2 y_2' + Q u_1 y_1 + Q u_2 y_2 = f(x)$$

or $$u_1[y_1'' + P\, y_1' + Q\, y_1] + u_2[y_2'' + P y_2' + Q y_2] + u_1' y_1' + u_2' y' = f(x)$$

Now making use of the relations

$$y_1'' + P\,(x)\, y_1' + Q\,(x)\, y_1 = 0$$

$$y_2'' + P\,(x)\, y_2' + Q\,(x)\, y_2 = 0$$

we get, $$u_1' y_1' + u_2' y_2' = f(x)$$

Hence u_1 and u_2 must be functions that satisfy the equations

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = f(x)$$

By using the Cramer's rule, the solution of this set of equations is given by

$$u_1' = \frac{W_1}{W}, u_2' = \frac{W_2}{W}$$

$a_1 x + b_1 y = c_1$

$a_2 x + b_2 y = c_2$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

u_1', u_2'

W_1

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

W

where $W, W_1 and W_2$ and denote the following determinants

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

The determinant W can be identified as the Wronskian of the solutions y_1 and y_2. Since the solutions y_1 and y_2 are linearly independent on *I*. Therefore

$$W(y_1(x), y_2(x)) \neq 0, \ \forall \ x \in I.$$

Now integrating the expressions for $u_1' and u_2'$ we obtain the value of u_1 and u_2, hence the particular solution of the non-homogeneous linear DE.

$y_p = u_1 y_1 + u_2 y_2$

Summary of the Method

To solve the second order non-homogeneous linear DE

$$a_2 y'' + a_1 y' + a_0 y = g(x),$$

using the variation of parameters, we need to perform the following steps:

Step 1

We find the complementary function by solving the associated homogeneous differential equation

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Step 2

If the complementary function of the equation is given by

$$y_c = c_1 y_1 + c_2 y_2$$

then y_1 and y_2 are two linearly independent solutions of the homogeneous differential equation. Computing Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Step 3

By dividing with a_2 , we transform the given non-homogeneous equation into the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

and we identify the function f(x).

Step 4

We now construct the determinants $W_1 and W_2$ given by

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 5

Next we determine the derivatives of the unknown variables $u_1 and u_2$ through the relations

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

Step 6

Integrate the derivatives $u_1' and u_2'$ to find the unknown variables $u_1 and u_2$. So that

$$u_1 = \int \frac{W_1}{W} \, dx, \quad u_2 = \int \frac{W_2}{W} \, dx$$

Step 7

Write a particular solution of the given non-homogeneous equation as

$$y_p = u_1 y_1 + u_2 y_2$$

Step 8

The general solution of the differential equation is then given by

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$$

Constants of Integration

We don't need to introduce the constants of integration, when computing the indefinite integrals in step 6 to find the unknown functions of $u_1 and u_2$. For, if we do introduce these constants, then

$$y_p = (u_1 + a_1) y_1 + (u_2 + b_1) y_2$$

So that the general solution of the given non-homogeneous differential equation is

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + (u_1 + a_1) y_1 + (u_2 + b_1) y_2$$

or

$$y = (c_1 + a_1) y_1 + (c_2 + b_1) y_2 + u_1 y_1 + u_2 y_2$$

If we replace $c_1 + a_1$ with C_1 and $c_2 + b_1$ with C_2, we obtain

$$y = C_1 y_1 + C_2 y_2 + u_1 y_1 + u_2 y_2$$

This does not provide anything new and is similar to the general solution found in step 8, namely

$$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$$

Example 1

Solve $$y'' - 4y' + 4y = (x+1)e^{2x}.$$

Solution:

Step 1

To find the complementary function

$$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad m^2 - 4m + 4 = 0$$

Put

$$y = e^{mx}, y' = me^{mx}, y'' = m^2 e^{mx}$$

Then the auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0, \Rightarrow m = 2,\ 2$$

Repeated real roots of the auxiliary equation

$$y_c = c_1 e^{2x} \quad + \quad c_2 x e^{2x}$$

Step 2

By the inspection of the complementary function y_c we make the identification

$$y_1 = e^{2x} \text{ and } y_2 = xe^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Therefore

$$W(y_1, y_2) = \left(e^{2x}, xe^{2x}\right) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x} \neq 0, \forall x$$

$\boxed{W = e^{4x}}$

Step 3

The given differential equation is

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Since this equation is already in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

Therefore, we identify the function as

$$f(x) = (x+1)e^{2x}$$

Step 4

We now construct the determinants

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Step 5

We determine the derivatives of the functions $u_1 and u_2$ in this step

$$\checkmark \quad u_1' = \frac{W_1}{W} = -\frac{(x+1)x\cancel{e^{4x}}}{\cancel{e^{4x}}} = -x^2 - x$$

Step 6

$$\checkmark \quad u_2' = \frac{W_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}} = x+1$$

Integrating the last two expressions, we obtain

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int (-x^2 - x)dx = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$u_2 = \int (x+1)\, dx \quad = \frac{x^2}{2} + x. \quad \checkmark$$

Remember!

We don’t have to add the constants of integration.

Step 7

Therefore, a particular solution of then given differential equation is

$$y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

or

$$y_p = \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

Step 8

Hence, the general solution of the given differential equation is

$$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

Example 2

Solve $\qquad 4y'' + 36y = \csc 3x.$

Solution:

Step 1

To find the complementary function we solve the associated homogeneous differential equation

$$4y'' + 36y = 0 \Rightarrow y'' + 9y = 0$$

$m^2 = -9$

The auxiliary equation is

$m = \pm\sqrt{-9}$

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

Roots of the auxiliary equation are complex. Therefore, the complementary function is

$m = \pm i3$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

Step 2

From the complementary function, we identify

$$y_1 = \cos 3x,\ \ y_2 = \sin 3x$$

as two linearly independent solutions of the associated homogeneous equation. Therefore

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

Step 3

By dividing with , we put the given equation in the following standard form

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

So that we identify the function as

$$f(x) = \frac{1}{4}\csc 3x$$

Step 4

We now construct the determinants $W_1 and W_2$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}\csc 3x \cdot \sin 3x = -\frac{1}{4}$$

$W = 3$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$= \frac{1}{4\sin^2 3x}$

Step 5

Therefore, the derivatives $u_1' and u_2'$ are given by

$$u_1' = \frac{W_1}{W} = -\frac{1}{12}, \quad u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

Step 6

Integrating the last two equations $w.r.t$, we obtain

$$u_1 = -\frac{1}{12}x \quad \text{and} \quad u_2 = \frac{1}{36}\ln|\sin 3x|$$

Here, no constants of integration are used.

Step 7

The particular solution of the non-homogeneous equation is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

Step 8

Hence, the general solution of the given differential equation is

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

Example 3

Solve $$y'' - y = \frac{1}{x}.$$

Solution:

Step 1

For the complementary function consider the associated homogeneous equation

$$y'' - y = 0$$

To solve this equation we put

$$y = e^{mx},\ y' = m\,e^{mx},\ y'' = m^2 e^{mx}$$

Then the auxiliary equation is:

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

The roots of the auxiliary equation are real and distinct. Therefore, the complementary function is

$$y_c = c_1 e^x + c_2 e^{-x}$$

Step 2

From the complementary function we find

$$y_1 = e^x, \quad y_2 = e^{-x}$$

The functions y_1 and y_2 are two linearly independent solutions of the homogeneous equation. The Wronskian of these solutions is

$$W\left(e^x,\ e^{-x}\right) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Step 3

The given equation is already in the standard form

$$y'' + p(x)y' + Q(x)y = f(x)$$

Here

$$f(x) = \frac{1}{x}$$

Step 4

We now form the determinants

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ 1/x & -e^{-x} \end{vmatrix} = -e^{-x}(1/x)$$

Step 5

$$W_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & 1/x \end{vmatrix} = e^{x}(1/x)$$

Therefore, the derivatives of the unknown functions and are given by

$$u_1' = \frac{W_1}{W} = -\frac{e^{-x}(1/x)}{-2} = \frac{e^{-x}}{2x}$$

$$u_2' = \frac{W_2}{W} = \frac{e^{x}(1/x)}{-2} = -\frac{e^{x}}{2x}$$

Step 6

We integrate these two equations to find the unknown functions $u_1 and u_2$

$$u_1 = \frac{1}{2}\int \frac{e^{-x}}{x}dx, u_2 = -\frac{1}{2}\int \frac{e^{x}}{x}dx$$

The integrals defining $u_1 and u_2$ cannot be expressed in terms of the elementary functions and it is customary to write such integral as:

$$u_1 = \frac{1}{2}\int_{x_\circ}^{x} \frac{e^{-t}}{t}\,dt, \quad u_2 = \frac{1}{2}\int_{x_\circ}^{x} \frac{e^{t}}{t}\,dt$$

Step 7

A particular solution of the non-homogeneous equations is

$$y_p = \frac{1}{2}e^{x}\int_{x_\circ}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_\circ}^{x} \frac{e^{t}}{t}\,dt$$

Step 8

Hence, the general solution of the given differential equation is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}e^{x}\int_{x_\circ}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_\circ}^{x} \frac{e^{t}}{t}\,dt$$

Problem

Solve the given differential equation by variation of parameters.

$$y'' - y = e^x + 1$$

Associated homg: $y'' - y = 0$

Aux. $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

$y_c = c_1 e^x + c_2 e^{-x}$

$\therefore y_1 = e^x, \; y_2 = e^{-x}$ are two of solutions of homg. eq.

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^{\cancel{x}-\cancel{x}} - e^{\cancel{-x}+\cancel{x}}$$

$$= -1 - 1 = -2$$

Since $y'' - y = e^{x} + 1$

Here $f(x) = e^{x} + 1$

Now, $W_1 = \begin{vmatrix} 0 & e^{-x} \\ e^{x}+1 & -e^{-x} \end{vmatrix} = 0 - e^{-x}(e^{x}+1)$

$= -e^{\cancel{-x+x}} - e^{-x}$

$$\boxed{W_1 = -1 - e^{-x}}$$

$W_2 = \begin{vmatrix} e^{x} & 0 \\ e^{x} & e^{x}+1 \end{vmatrix} = e^{x}(e^{x}+1) - 0 = e^{2x} + e^{x}$

$u_1' = \dfrac{W_1}{W}, \quad u_2' = \dfrac{W_2}{W}$

$u_1' = \dfrac{-1-e^{-x}}{-2} = \dfrac{1+e^{-x}}{2}$

$u_1 = \dfrac{1}{2}\left[x - e^{-x}\right]$

$$u_2' = \frac{W_2}{W} = \frac{e^{2x} + e^{x}}{-2} = -\frac{1}{2}\left(e^{2x} + e^{x}\right)$$

Integrating

$$u_2 = -\frac{1}{2}\left[\frac{e^{2x}}{2} + e^{x}\right] = -\frac{1}{4}\left[e^{2x} + 2e^{x}\right]$$

$$\therefore\ y_p = \frac{1}{2}\left[x - e^{-x}\right]e^{x} + \left(-\frac{1}{4}\right)\left[e^{2x} + 2e^{x}\right]e^{-x}$$

The general sol. of the DE is

$$\therefore\ y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}\left[x - e^{-x}\right]e^{x} - \frac{1}{4}\left[e^{2x} + 2e^{x}\right]e^{-x}$$

For practice solve the problems from
Exercise 4.6
of your text book

$x y' + x^2 y = x^3$

Summary

- Motivation for the method of variation of parameters.
- Method for first order DEs.
- Method for second order DEs.
- Summary of the method.
- No need to add the constants.
- Some examples.

$\boxed{\frac{1}{x}, \quad \operatorname{cosec} x}$