Solution of Differential Equations

Topics to be discussed

- Solution of Differential equations
- Power series method
- Bessel's equation by Frobenius method

POWER SERIES METHOD

Power series method

- Method for solving linear differential equations with variable co-efficient.

$$y'' + p(x)y' + q(x)y = f(x)$$

p(x) and q(x) are variable co-effecients.

Power series method (Cont'd)

- Solution is expressed in the form of a power series.

$$Let \qquad y=\sum_{m=0}^{\infty}a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots$$

$$so, \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots\dots\dots$$

$$y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 \dots\dots\dots$$

Power series method (Cont'd)

- Substitute $y, y^{'}, y^{''}$
- Collect like powers of x
- Equate the sum of the co-efficients of each occurring power of x to zero.
- Hence the unknown co-efficients can be determined.

Examples

Example 1: Solve $y' - y = 0$

Example 2: Solve $y' = 2xy$

Example 3: Solve $y'' + y = 0$

Power series method (Cont'd)

- The general representation of the power series solution is,

$$y = \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots\ldots$$

Theory of power series method

- ***Basic concepts***

A power series is an infinite series of the form

(1) $$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots\ldots$$

x is the variable, the **center** x_0, and the **coefficients** a_0, a_1, a_2 are real.

Theory of power series method (Cont'd)

(2) The nth partial sum of (1) is given as,

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ... + a_n(x - x_0)^n$$

where n=0,1,2..

(3) The remainder of (1) is given as,

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + ...$$

Convergence Interval. Radius of convergence

- Theorem: Let $y=\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ...$ be a power series ,then there exists some number ∞≤R≤0 ,called its **radius of convergence** such that the series is convergent for $|(x-x_0)| < R$ and divergent for $|(x-x_0)| > R$

- The values of x for which the series converges, form an interval, called the **convergence interval**

Convergence Interval. Radius of convergence

What is R?

The number R is called the **radius of convergence**. It can be obtained as,

$$R = 1 \Big/ \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right|$$

or

$$R = 1 \Big/ \lim_{m\to\infty} \sqrt[m]{|a_m|}$$

Examples (Calculation of R)

Example 1: $$\sum \frac{nx^n}{2^{n+1}}$$

Answer

$$\frac{|(n+1)term|}{|(n)term|} = \frac{(n+1)|x^{n+1}|}{2^{n+2}} * \frac{2^{n+1}}{(n)|x^n|} = \frac{(n+1)|x|}{2.n}$$

As n→∞, $$\frac{(n+1)|x|}{2.n} \to \frac{|x|}{2}$$

Examples (Calculation of R) Cont'd

Hence $\quad R = 2$

- The given series converges for

$$\left|(x-0)\right| < 2$$

- The given series diverges for

$$\left|(x-0)\right| > 2$$

Examples

- Example 2:

  Find the radius of convergence of the following series,

$$\sum_{n=0}^{\infty} \frac{x^n}{n.2^n}$$

Examples

- Example 3:

  Find the radius of convergence of the following series,

$$\sum_{n=0}^{\infty} \frac{x^n}{n.(n-1).(n-2).(n-3)...1}$$

Operations of Power series: Theorems

(1) **Equality of power series**

If $\sum_{n=0}^{\infty} a_n (x - x_a)^n = \sum_{n=0}^{\infty} b_n (x - x_a)^n,$ with $R \neq 0$

then $a_n = b_n$,for all n

Corollary

If $\sum_{n=0}^{\infty} a_n (x - x_a) = 0,$ all a_n=0, for all n ,R>0

Theorems (Cont'd)

(2) **Termwise Differentiation**

If $y = \sum_{n=0}^{\infty} a_n x^n$ is convergent, then derivatives involving y(x) such as $y^{'}(x), y^{''}(x), etc$ are also convergent.

(3) **Termwise Addition**

If $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent in the same domain x, then the sum also converges in that domain.

Existence of Power Series Solutions. Real Analytic Functions

- The **power series solution will exist** and be **unique** provided that the variable co-efficients **p(x), q(x), and f(x) are analytic in the domain of interest.**

What is a real analytic function ?

A real function f(x) is called analytic at a point $x=x_0$ if it ***can be represented by a power series in powers of $x-x_0$ with radius of convergence $R>0$.***

Example

- Let's try this

$$y''(x) + xy'(x) + y(x) = 0$$

BESSEL'S EQUATION.

BESSEL FUNCTIONS $J_\nu(x)$

Application

- Heat conduction
- Fluid flow
- Vibrations
- Electric fields

Bessel's equation

- Bessel's differential equation is written as

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

or in standard form,

$$y'' + \frac{1}{x}y' + (1 - \frac{n^2}{x^2})y = 0$$

Bessel's equation (Cont'd)

- n is a non-negative real number.
- x=0 is a regular singular point.
- The Bessel's equation is of the type,

$$y'' + \frac{p(x)}{r(x)} y' + \frac{q(x)}{r(x)} y = \frac{f(x)}{r(x)}$$

and is solved by the Frobenius method

Non-Analytic co-efficients –Methods of Frobenius

- If x is not analytic, it is a singular point.

$$r(x)y'' + p(x)y' + q(x)y = f(x)$$

$$\rightarrow \qquad y'' + \frac{p(x)}{r(x)}y' + \frac{q(x)}{r(x)}y = \frac{f(x)}{r(x)}$$

The points where r(x)=0 are called as singular points.

Non-Analytic co-efficients –Methods of Frobenius (Cont'd)

- The solution for such an ODE is given as,

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

Substituting in the ODE for values of y(x),

$y'(x), y''(x)$, equating the co-efficient of x^m and obtaining the roots gives the indical solution

Non-Analytic co-efficients –Methods of Frobenius (Cont'd)

We solve the Bessel's equation by Frobenius method.

Substituting a series of the form,

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Indical solution

$$r_1 = n(n \geq 0) \qquad r_2 = -n$$

General solution of Bessel's equation

- Bessel's function of the first kind of order n

is given as,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

Substituting –n in place of n, we get

$$J_{-n}(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n} m!(-n+m+1)!}$$

Example

- Compute

$$J_0(x)$$

- Compute

$$J_1(x)$$

General solution of Bessel's equation

- If n is not an integer, a general solution of Bessel's equation for all x≠0 is given as

$$y(x) = \left[c_1.J_n(x) + c_2.J_{-n}(x)\right]$$

Properties of Bessel's Function

$J_0(0) = 1$

$J_n(x) = 0 \quad (n>0)$

$J_{-n}(x) = (-1)^n J_n(x)$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$\frac{d}{dx}\left[x^{n} J_n(x)\right] = -x^{n} J_{n-1}(x)$$

Properties of Bessel's Function (Cont'd)

$$\frac{d}{dx}\left[J_n(x)\right]=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$$

$$\left[x.J_{n+1}(x)\right]=\left[2.n.J_n(x)-x.J_{n-1}(x)\right]$$

$$\int (x^n J_{n-1}(x))dx = x^n J_n(x)+C$$

$$\int (x^{-n} J_{n+1}(x))dx = -x^{-n} J_n(x)+C$$

Properties of Bessel's Function (Cont'd)

$$\left[J_{n-1}(x) - J_{n+1}(x)\right] = 2.J_{n}^{'}(x)$$

$$\left[J_{n-1}(x) + J_{n+1}(x)\right] = \frac{2.n}{x} J_{n}(x)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Examples

- Example : Using the properties of Bessel's functions compute,

1. $J_3(x)$

2. $\int (x^{-2} J_2(x))dx$

3. $J_{3/2}(x)$