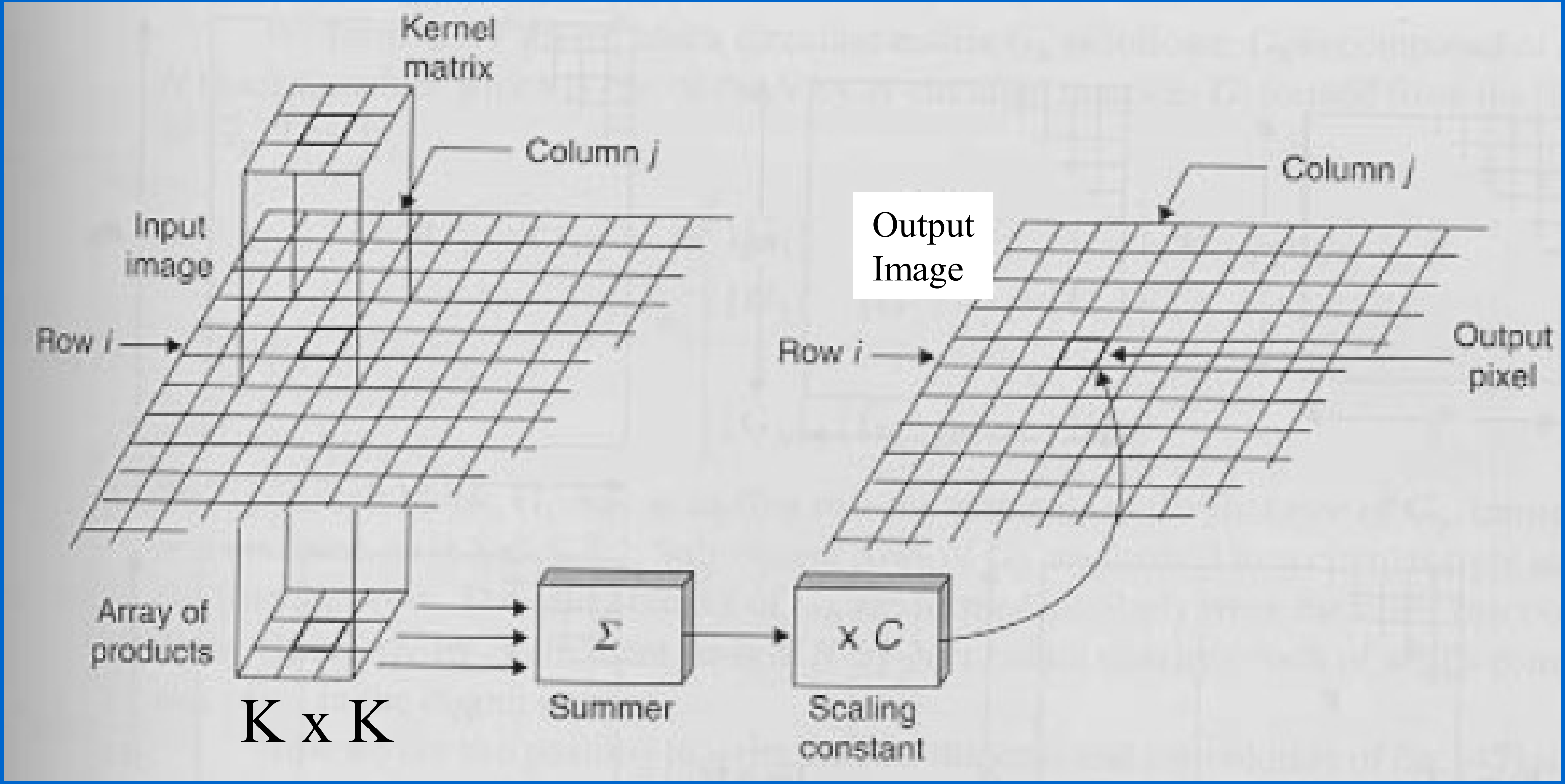
Convolution

Correlation - Review



$$g(x,y) = w(x,y) \bullet f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x+s, y+t)$$

Convolution - Review

- Same as correlation except that the mask is **flipped** both horizontally and vertically.

$$g(x,y) = w(x,y) * f(x,y) = \sum_{s=-K/2}^{K/2} \sum_{t=-K/2}^{K/2} w(s,t) f(x-s, y-t)$$

- Note that if w(x,y) is symmetric, that is w(x,y)=w(-x,-y), then convolution is equivalent to correlation!

1D Continuous Convolution - Definition

- Convolution is defined as follows:

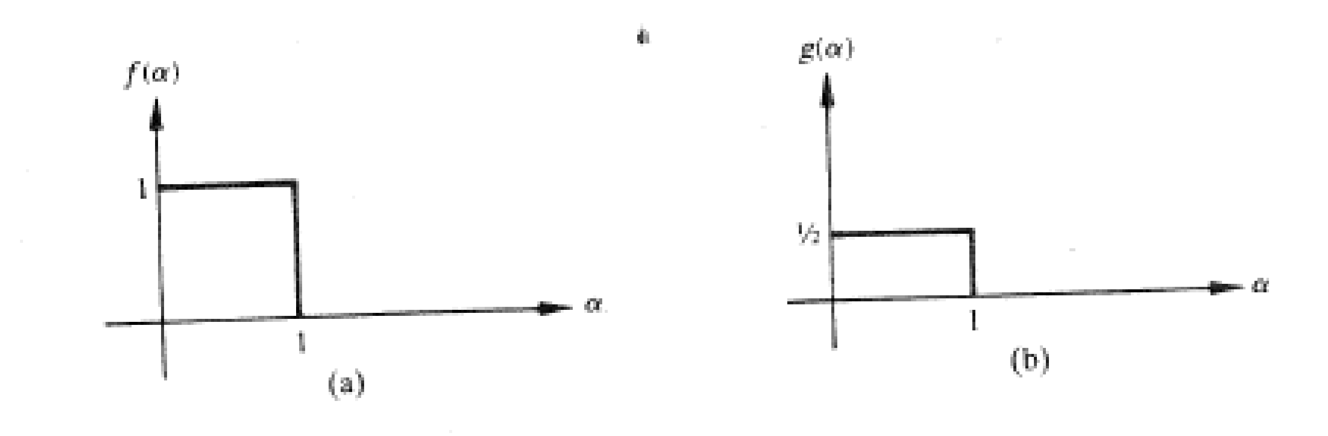
$$f(x) * g(x) = \int_{-\infty}^{\infty} f(a)g(x-a)da$$

- Convolution is commutative

$$f(x) * g(x) = g(x) * f(x)$$

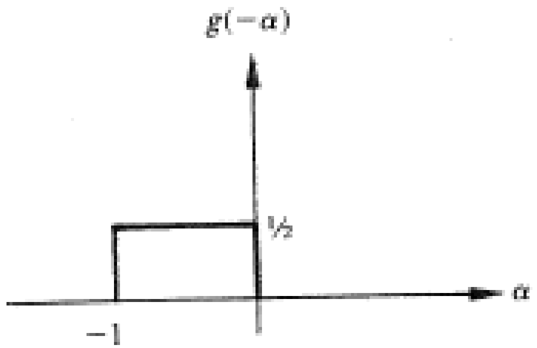
Example

- Suppose we want to compute the convolution of the following two functions:



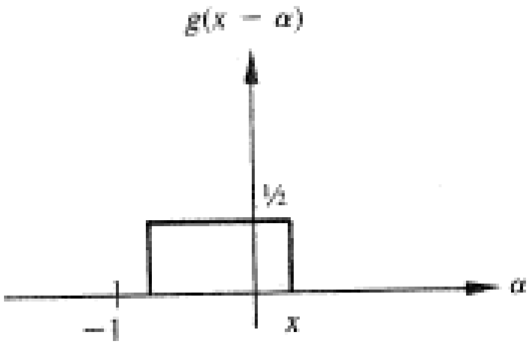
Example (cont'd)

Step1: find $g(-a)$



(c)

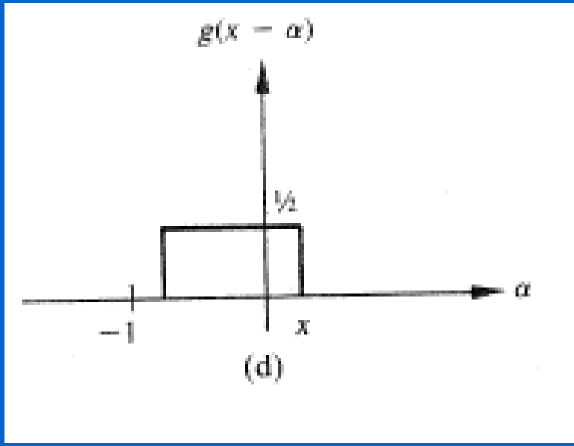
Step2: find $g(x-a)$

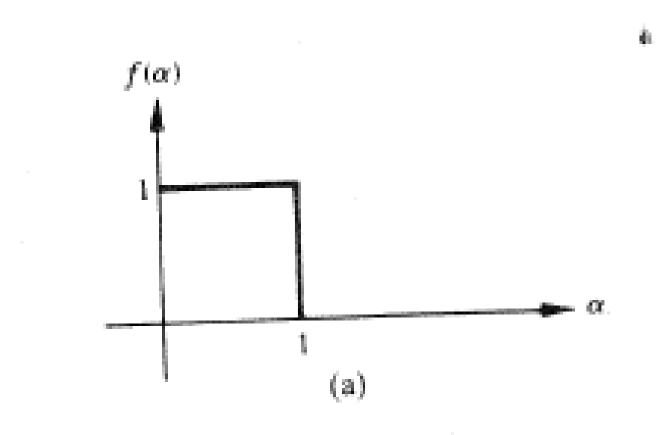


(d)

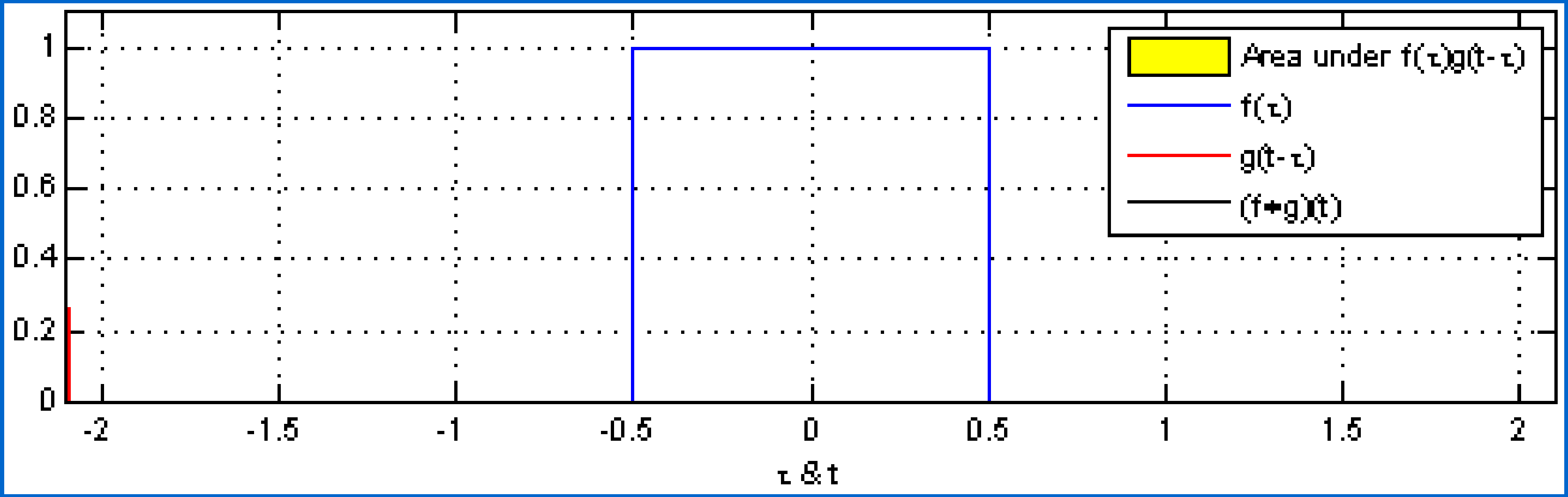
Example (cont'd)

Step 3:





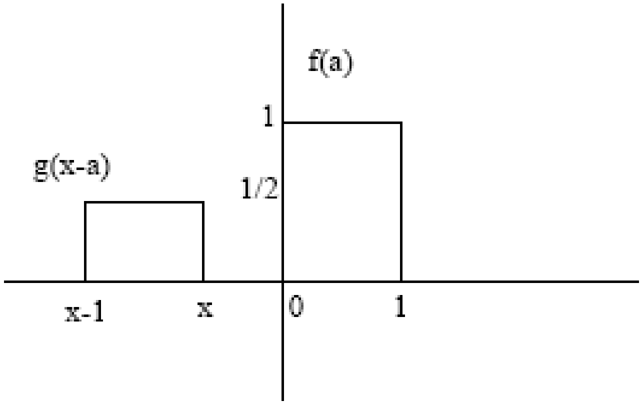
Example (cont'd)



Example (cont'd)

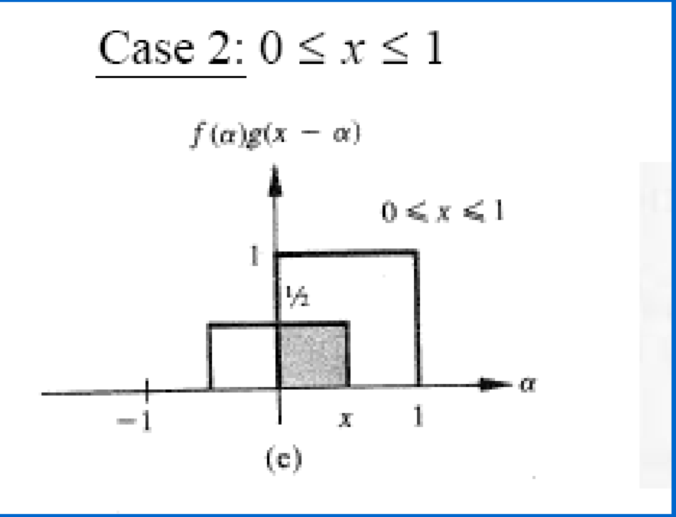
Step 3: consider all possible cases for x:

Case 1: $x < 0$



no overlap: $\int_{-\infty}^{\infty} f(a)g(x-a)da = 0$

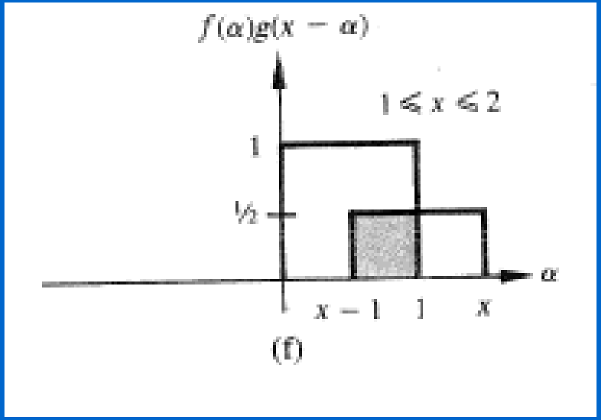
Example (cont'd)



$$\int_{-\infty}^{\infty} f(a)g(x-a)da = \int_{0}^{x} 1\,\frac{1}{2}\,da = \frac{x}{2}$$

Example (cont'd)

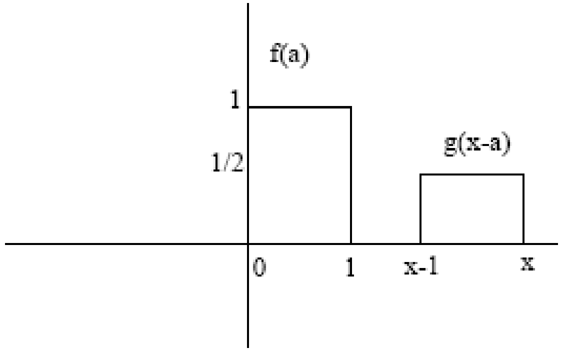
Case 3: $1 \leq x \leq 2$



$$\int_{-\infty}^{\infty} f(a)g(x-a)da = \int_{x-1}^{1} 1\,\frac{1}{2}\,da = 1 - \frac{x}{2}$$

Example (cont'd)

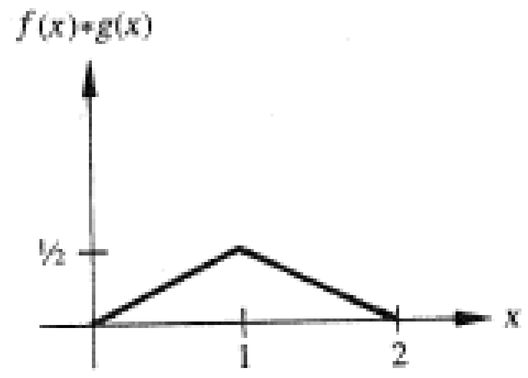
Case 4: $x > 2$



no overlap: $\int_{-\infty}^{\infty} f(a)g(x-a)da = 0$

Example (cont'd)

$$f(x) * g(x) = \begin{cases} x/2 & 0 \le x \le 1 \\ 1 - x/2 & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$$

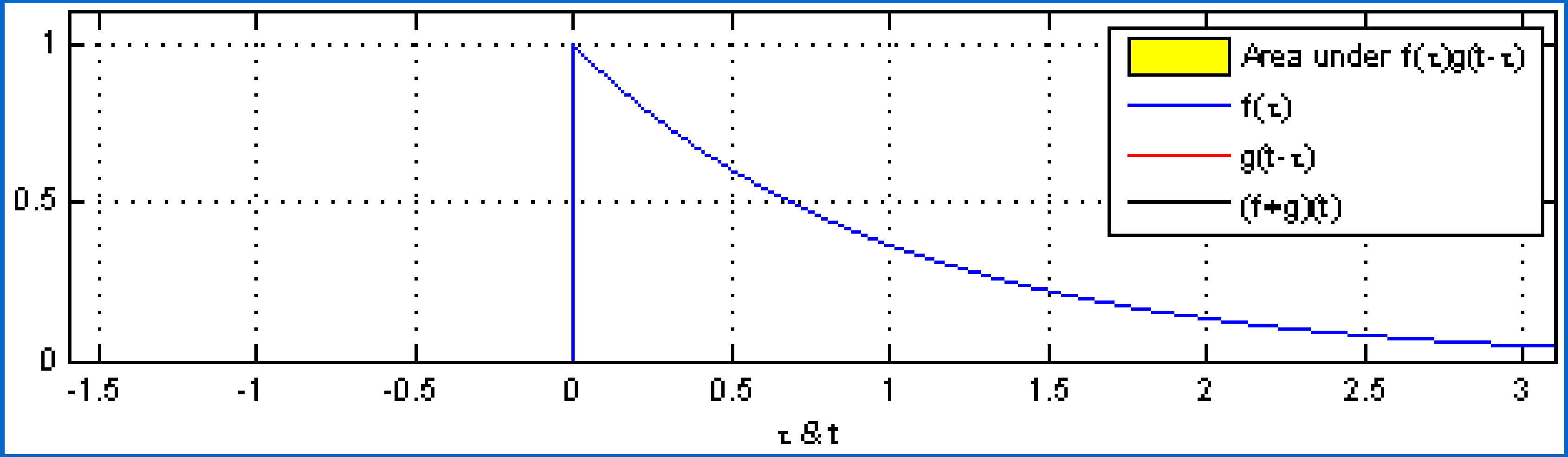


(g)

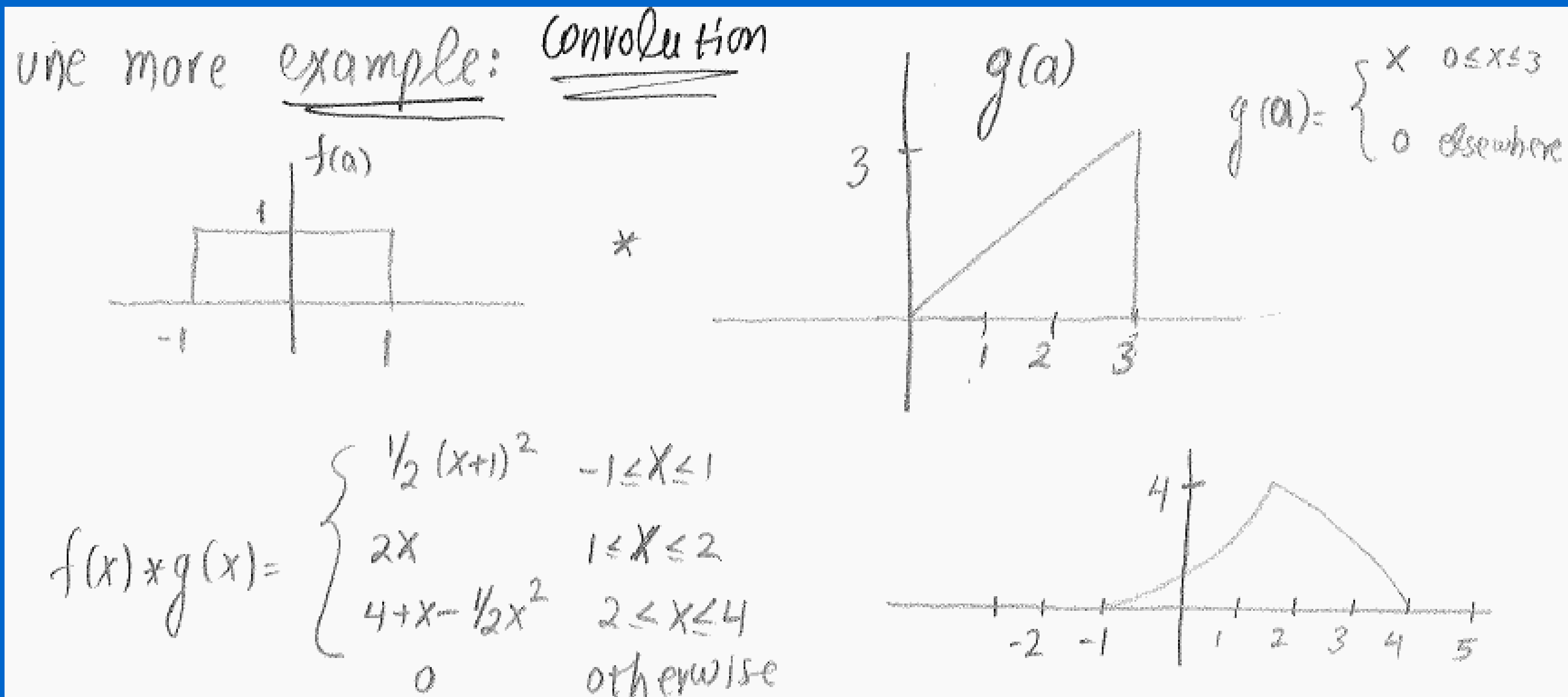
Important Observations

- The extent of f(x) * g(x) is equal to the extent of f(x) plus the extent of g(x)
- For every x, the limits of the integral are determined as follows:
  - Lower limit: MAX (left limit of f(x), left limit of g(x-a))
  - Upper limit: MIN (right limit of f(x), right limit of g(x-a))

Example (cont'd)



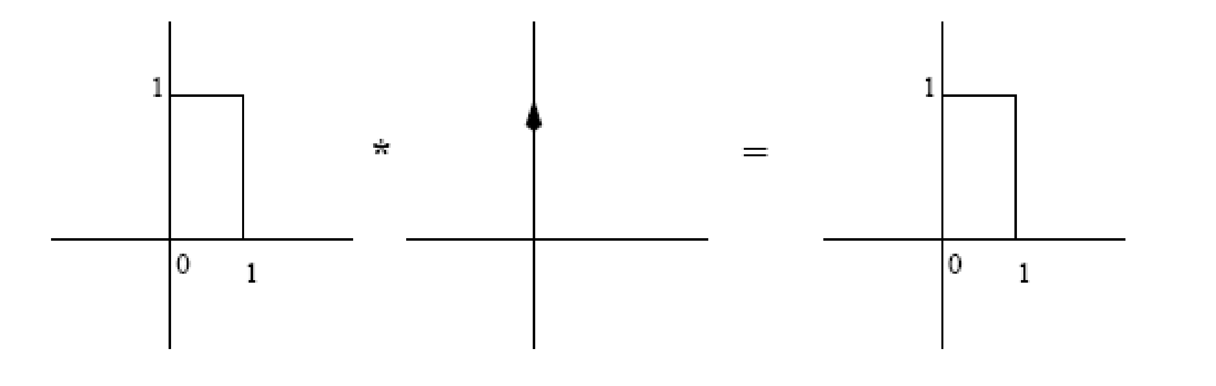
Example


one more example: Convolution
f(a)
1
-1
1
*
g(a)
3
1 2 3
g(a)= { x 0≤x≤3
0 elsewhere
f(x)*g(x)= { 1/2 (x+1)^2 -1≤x≤1
2x 1≤x≤2
4+x-1/2x^2 2≤x≤4
0 otherwise
4
-2 -1 1 2 3 4 5

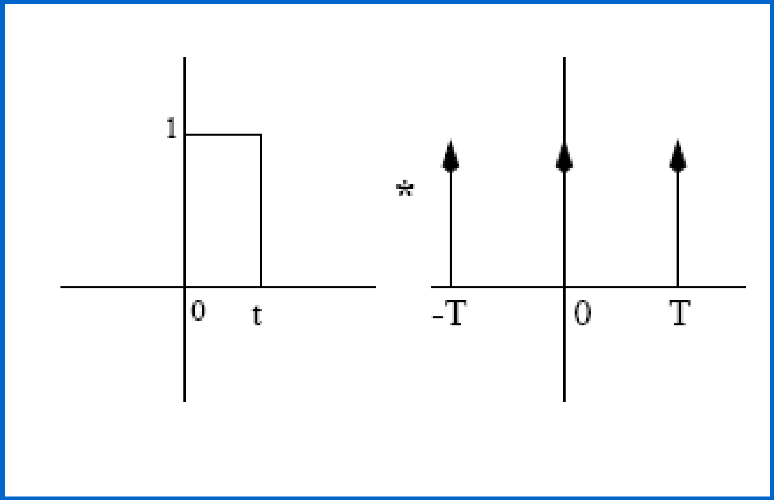
Convolution with an impulse (i.e., delta function)

$$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(a)\delta(x-a)da = f(x)$$

(since $\delta(x-a) = 1$ if $a = x$)



Convolution with an "train" of impulses



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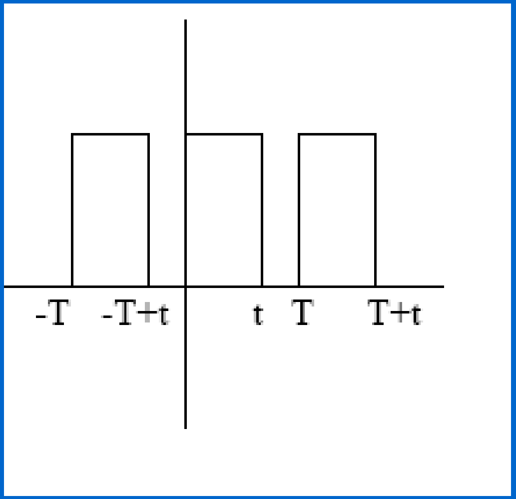


Convolution Theorem

- Convolution in the time domain is equivalent to multiplication in the frequency domain.

$$f(x) * g(x) <—> F(u)G(u)$$

$$f(x) \longleftrightarrow F(u)$$
$$g(x) \longleftrightarrow G(u)$$

- Multiplication in the time domain is equivalent to convolution in the frequency domain.

$$f(x)g(x) <—> F(u) * G(u)$$

Efficient computation of (f * g)

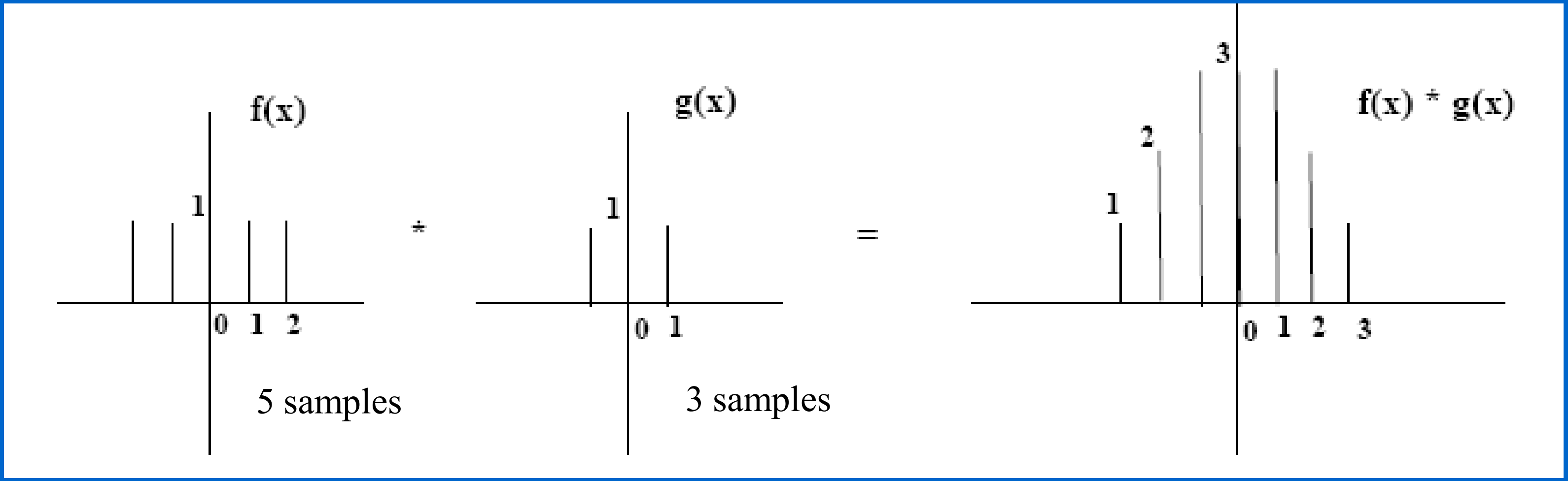
- 1. Compute $\mathcal{F}(f(x)) = F(u)$ and $\mathcal{F}(g(x)) = G(u)$
- 2. Multiply them: $F(u)G(u)$
- 3. Compute the inverse FT: $\mathcal{F}^{-1}(F(u)G(u)) = f(x) * g(x)$

Discrete Convolution

- Replace integral with summation
- Integration variable becomes an index.
- Displacements take place in discrete increments

$$f(x) * g(x) = \sum_{m=-\infty}^{\infty} f(m) g(x-m), \; -\infty < x < \infty$$

Discrete Convolution (cont'd)



(length of $f * g$ = length of f + length of g - 1)

Convolution Theorem in Discrete Case

- Input sequences:

$$\{f(0), f(1), \ldots, f(A-1)\}, \{g(0), g(1), \ldots, g(B-1)\}$$

- Length of output sequence:

$$M = A + B - 1$$

- Extended input sequences (i.e., pad with zeroes)

$$f_e(x) = \begin{cases} f(x) & 0 \leq x \leq A-1 \\ 0 & A \leq x \leq M-1 \end{cases} \qquad g_e(x) = \begin{cases} g(x) & 0 \leq x \leq B-1 \\ 0 & B \leq x \leq M-1 \end{cases}$$

Convolution Theorem in Discrete Case (cont'd)

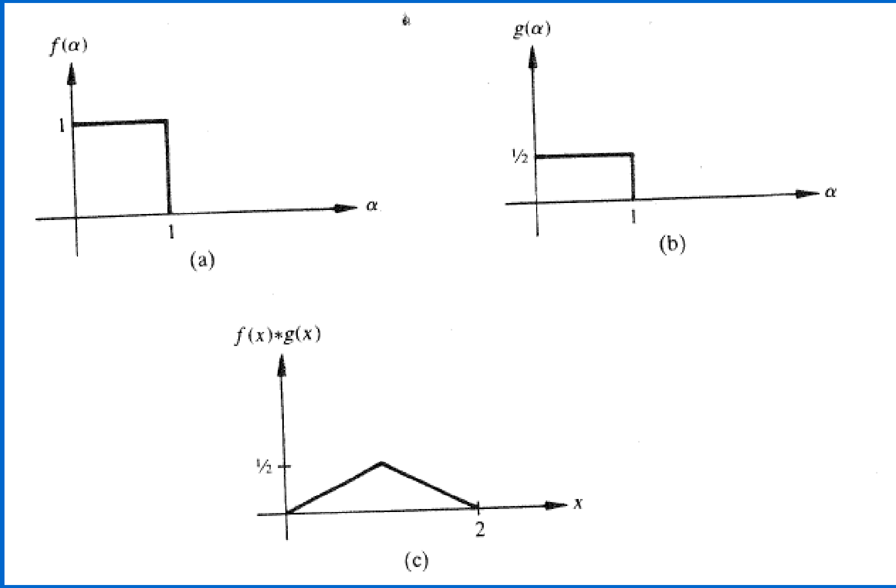
- When dealing with discrete sequences, the convolution theorem holds true for the extended sequences **only**, i.e., $(M \geq A + B - 1)$

$$f_e(x) * g_e(x) \Longleftrightarrow F_e(u)\, G_e(u)$$

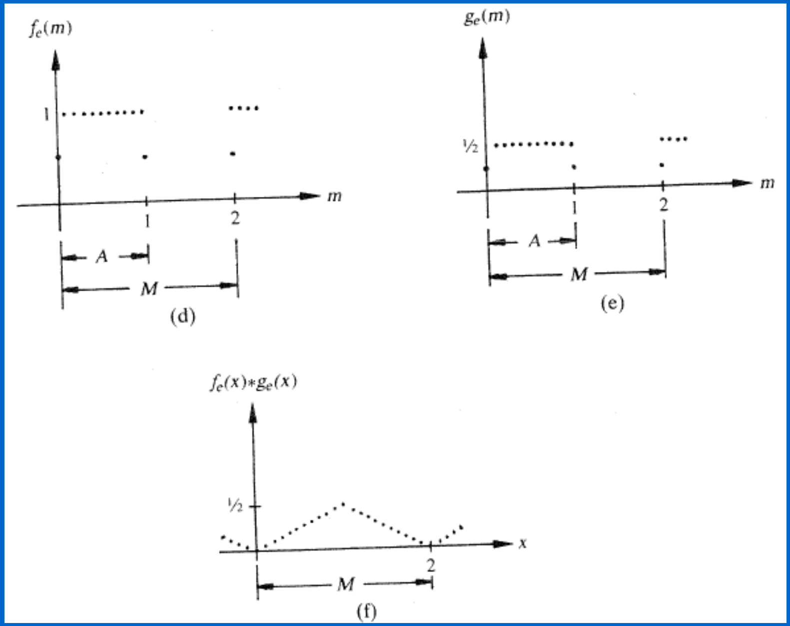
$$\text{where } f_e(x) * g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

Why?

continuous case



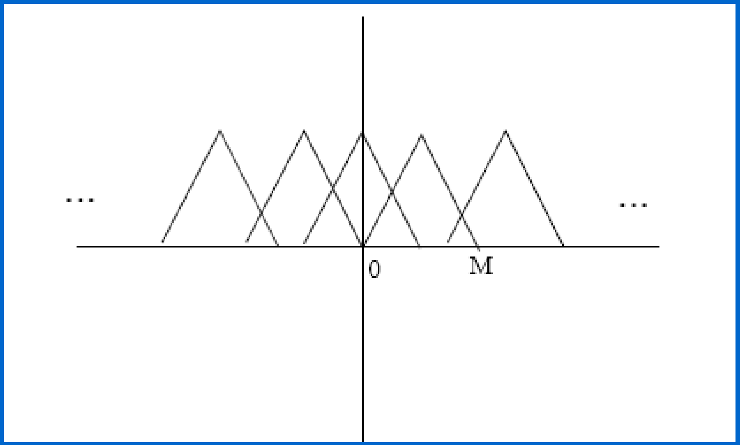
discrete case



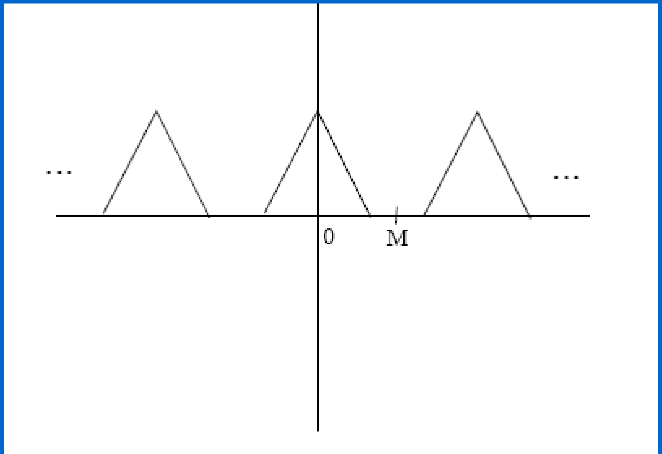
$f_e(x) * g_e(x)$

Using DFT, it will be a periodic function with period M (since DFT is periodic)

Why? (cont'd)



If **M<A+B-1**, the periods will overlap



If **M>=A+B-1**, the periods will not overlap

2D Convolution

- Definition

$$f(x,y) * g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a,b) g(x-a, y-b) \, da \, db$$

- 2D convolution theorem

$$f(x,y) * g(x,y) < --- > F(u,v)\ G(u,v)$$

$$f(x,y)\ g(x,y) < --- > F(u,v) * G(u,v)$$

Discrete 2D convolution

- Suppose f(x,y) and g(x,y) are images of size A x B and C x D
- The size of f(x,y) * g(x,y) would be N x M where N=A+C-1 and M=B+D-1
- Extended images (i.e., pad with zeroes):

$$f_e(x,y) = \begin{cases} f(x,y) & 0 \le x \le A-1 \text{ and } 0 \le y \le B-1 \\ 0 & A \le x \le M-1 \text{ and } B \le y \le N-1 \end{cases}$$

$$g_e(x) = \begin{cases} g(x,y) & 0 \le x \le C-1 \text{ and } 0 \le y \le D-1 \\ 0 & C \le x \le M-1 \text{ and } D \le y \le N-1 \end{cases}$$

Discrete 2D convolution (cont'd)

- The convolution theorem holds true for the extended images.

$$f_e(x,y) * g_e(x,y) <\!\!-\!\!-\!\!> F_e(u,v)G_e(u,v)$$

$$f_e(x,y) * g_e(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) g_e(x-m, y-n)$$

$$(x = 0, 1, \ldots, M-1,\ y = 0, 1, \ldots, N-1)$$