Fourier Series

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- Fourier Series
- Complex Form of the Fourier Series
- Impulse Train
- Analysis of Periodic Waveforms
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Fourier Series

Periodic Functions

The Mathematic Formulation

- Any function that satisfies

$$f(t) = f(t + T)$$

where T is a constant and is called the *period* of the function.

Example:

$$f(t) = \cos\frac{t}{3} + \cos\frac{t}{4} \qquad \text{Find its period.}$$

$$f(t) = f(t+T) \implies \cos\frac{t}{3} + \cos\frac{t}{4} = \cos\frac{1}{3}(t+T) + \cos\frac{1}{4}(t+T)$$

Fact: $\cos\theta = \cos(\theta + 2m\pi)$

$$\frac{T}{3} = 2m\pi \qquad\qquad T = 6m\pi$$

$$\frac{T}{4} = 2n\pi \qquad\qquad T = 8n\pi \qquad\qquad T = 24\pi \quad \text{smallest } T$$

Example:

$$f(t) = \cos \omega_1 t + \cos \omega_2 t \quad \text{Find its period.}$$

$$f(t) = f(t+T) \implies \cos \omega_1 t + \cos \omega_2 t = \cos \omega_1 (t+T) + \cos \omega_2 (t+T)$$

$$\omega_1 T = 2m\pi$$

$$\omega_2 T = 2n\pi$$
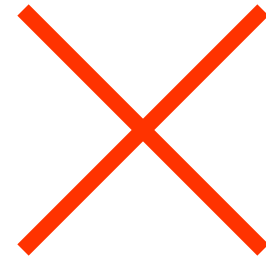
$$\implies \frac{\omega_1}{\omega_2} = \frac{m}{n} \implies \frac{\omega_1}{\omega_2} \text{ must be a rational number}$$

Example:

$$f(t) = \cos 10t + \cos(10 + \pi)t$$
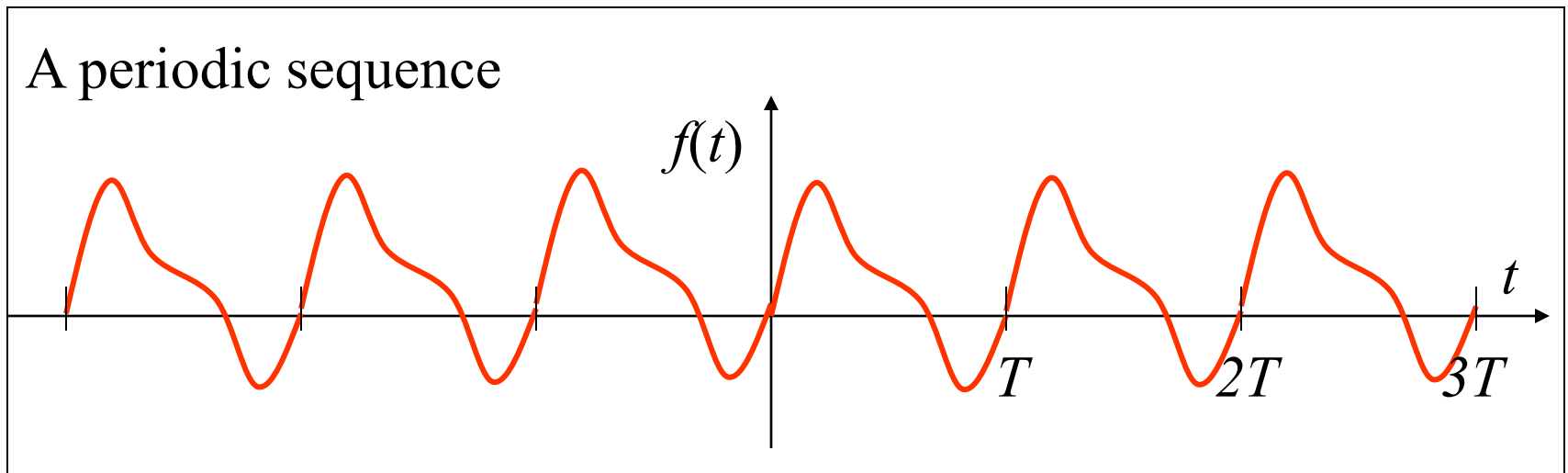
Is this function a periodic one?

$$\frac{\omega_1}{\omega_2} = \frac{10}{10 + \pi} \qquad \text{not a rational number}$$

Fourier Series

Fourier Series

Introduction

- Decompose a periodic input signal into *primitive periodic components*.

A periodic sequence

Synthesis

$$f(t) = \frac{a_0}{2} + \underbrace{\sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T}}_{} + \underbrace{\sum_{n=1}^{\infty} b_n \sin \frac{2\pi nt}{T}}_{}$$

DC Part Even Part Odd Part

T is a period of all the above signals

Let $\omega_0 = 2\pi/T$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Orthogonal Functions

- Call a set of functions $\{\phi_k\}$ *orthogonal* on an interval $a < t < b$ if it satisfies

$$\int_a^b \phi_m(t)\phi_n(t)dt = \begin{cases} 0 & m \neq n \\ r_n & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi/T$.

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$ We now prove this one

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

Proof

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt \qquad m \neq n$$

$$= \frac{1}{2}\int_{-T/2}^{T/2}\cos[(m+n)\omega_0 t]dt + \frac{1}{2}\int_{-T/2}^{T/2}\cos[(m-n)\omega_0 t]dt$$

$$= \frac{1}{2}\frac{1}{(m+n)\omega_0}\sin[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2} + \frac{1}{2}\frac{1}{(m-n)\omega_0}\sin[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{2}\frac{1}{(m+n)\omega_0}\underbrace{2\sin[(m+n)\pi]}_{0} + \frac{1}{2}\frac{1}{(m+n)\omega_0}\underbrace{2\sin[(m-n)\pi]}_{0}$$

$$= 0$$

Proof

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\cos^2\alpha = \frac{1}{2}[1+\cos 2\alpha]$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt \qquad m = n$$

$$= \int_{-T/2}^{T/2} \cos^2(m\omega_0 t)dt \;\; = \frac{1}{2}\int_{-T/2}^{T/2}[1+\cos 2m\omega_0 t]dt$$

$$= \frac{1}{2}t\Big|_{-T/2}^{T/2} + \underbrace{\frac{1}{4m\omega_0}\sin 2m\omega_0 t]\Big|_{-T/2}^{T/2}}_{0}$$

$$= \frac{T}{2}$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

an orthogonal set.

$$\left\{ \begin{array}{l} 1, \\ \cos\omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \cdots \\ \sin\omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \cdots \end{array} \right\}$$

Define $\omega_0 = 2\pi/T$.

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

Decomposition

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt \qquad n = 1, 2, \cdots$$

Proof

Use the following facts:

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0, \quad \text{for all } m \text{ and } n$$

Example (Square Wave)
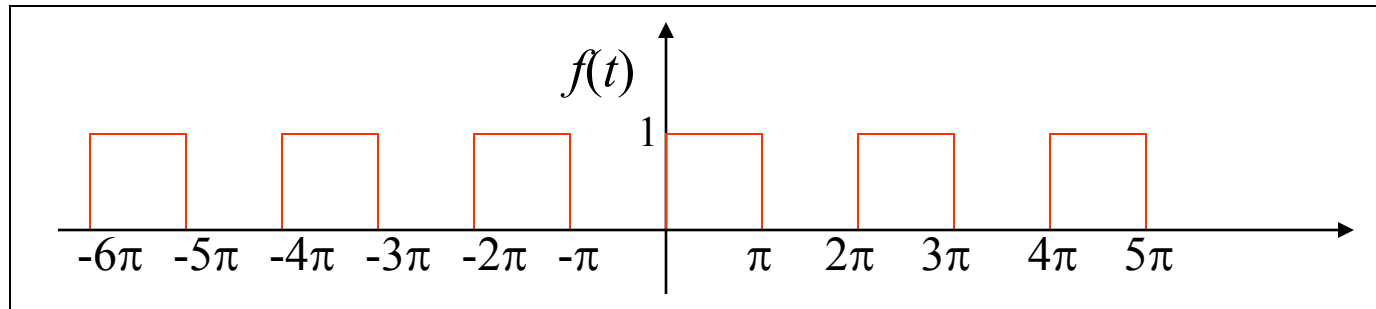
$$a_0 = \frac{2}{2\pi} \int_0^\pi 1 dt = 1$$

$$a_n = \frac{2}{2\pi} \int_0^\pi \cos nt \, dt = \frac{1}{n\pi} \sin nt \Big|_0^\pi = 0 \quad n = 1, 2, \cdots$$

$$b_n = \frac{2}{2\pi} \int_0^\pi \sin nt \, dt = -\frac{1}{n\pi} \cos nt \Big|_0^\pi = -\frac{1}{n\pi} (\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

Example (Square Wave)

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$
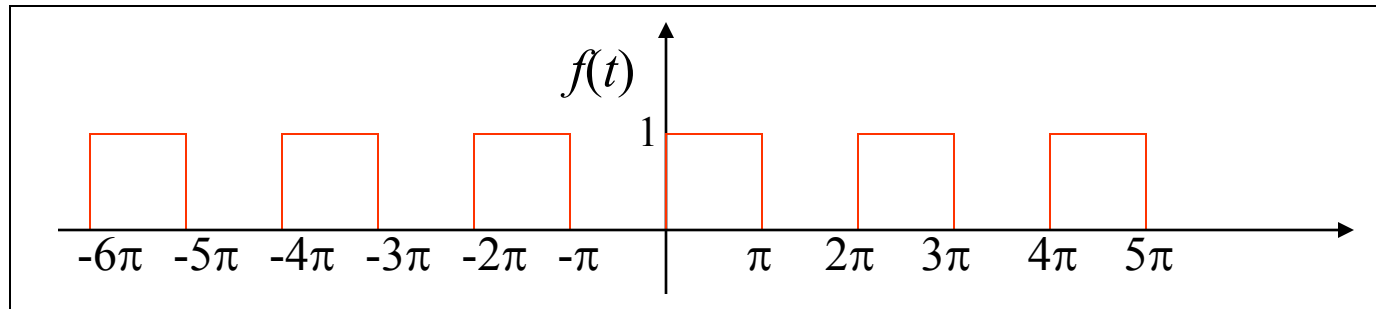
$$a_0 = \frac{2}{2\pi}\int_0^\pi 1\,dt = 1$$

$$a_n = \frac{2}{2\pi}\int_0^\pi \cos nt\,dt = \frac{1}{n\pi}\sin nt\Big|_0^\pi = 0 \qquad n = 1,2,\cdots$$

$$b_n = \frac{1}{2\pi}\int_0^\pi \sin nt\,dt = -\frac{1}{n\pi}\cos nt\Big|_0^\pi = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots \right)$$

$$a \quad \frac{2}{\pi}\int^{\pi} 1\, dt = 1$$

a

b

$n = 1,3,5,\cdots$

$n = 2,4,6,\cdots$
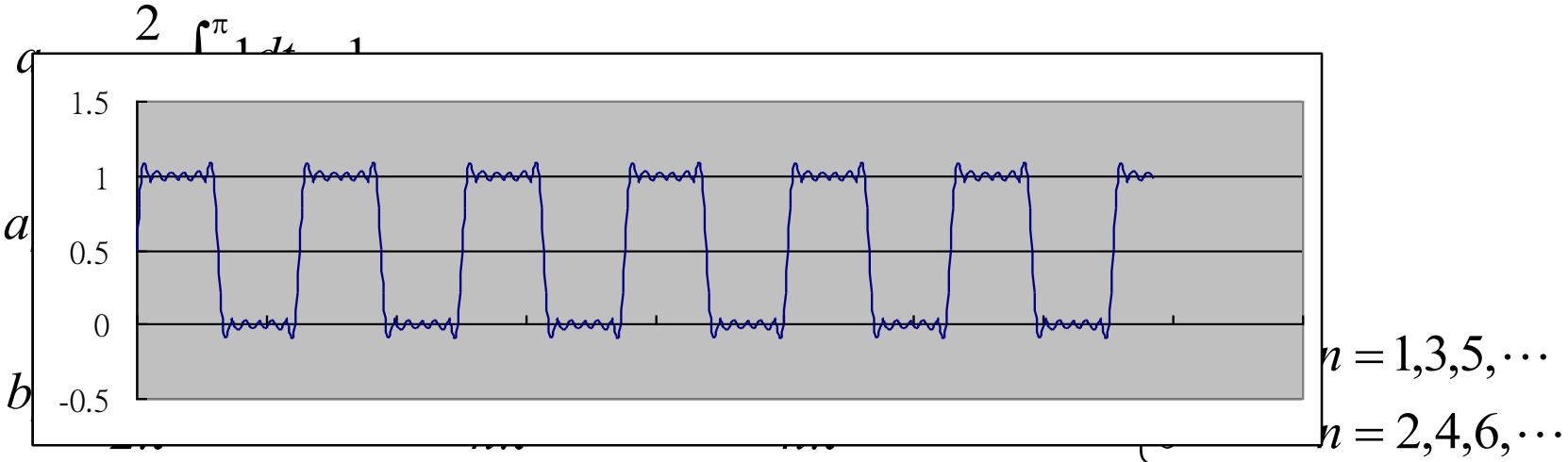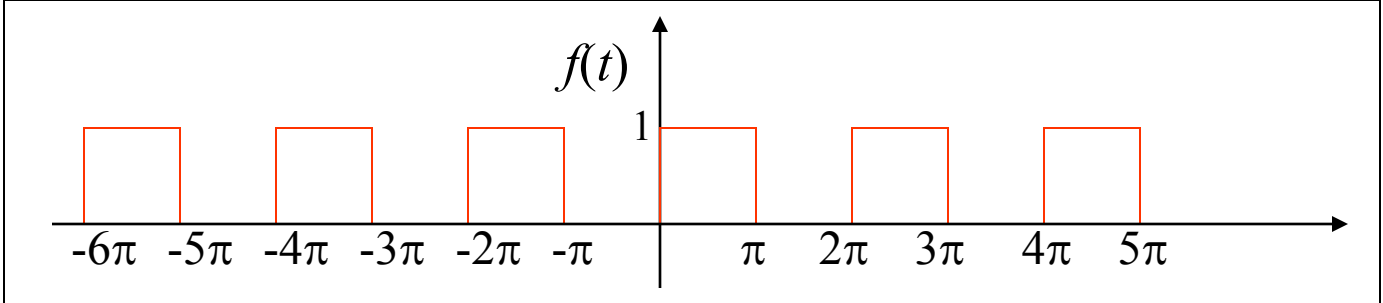
Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

DC Part　　Even Part　　Odd Part

T is a period of all the above signals

Harmonics

Define $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$, called the *fundamental angular frequency*.

Define $\omega_n = n\omega_0$, called the *n*-th *harmonic* of the periodic function.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$
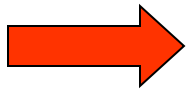
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \omega_n t \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\cos \theta_n \cos \omega_n t + \sin \theta_n \sin \omega_n t \right)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

Amplitudes and Phase Angles

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

harmonic amplitude   phase angle

$$C_0 = \frac{a_0}{2}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

Fourier Series

Complex Form of the Fourier Series

Complex Exponentials

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j \sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

Complex Form of the Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - \frac{j}{2}\sum_{n=1}^{\infty} b_n\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right]$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = c_0 + \sum_{n=1}^{\infty} \left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} \right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{T} \left[\int_{-T/2}^{T/2} f(t) \cos n\omega_0 t \, dt - j \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt \right]$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j \sin n\omega_0 t) dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{jn\omega_0 t} dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$
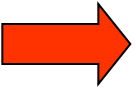
$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

If $f(t)$ is real,

$$\Rightarrow \quad c_{-n} = c_n^*$$

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$n = \pm 1, \pm 2, \pm 3, \cdots$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$$c_0 = \frac{1}{2}a_0$$

Complex Frequency Spectra

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) \qquad n = \pm 1, \pm 2, \pm 3, \cdots$$

$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$c_0 = \frac{1}{2} a_0$$
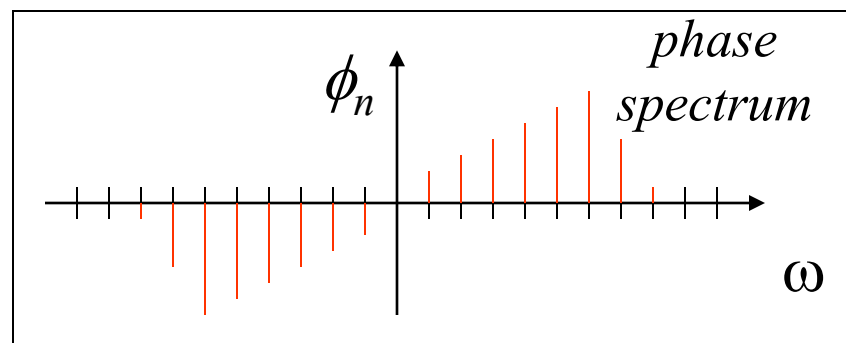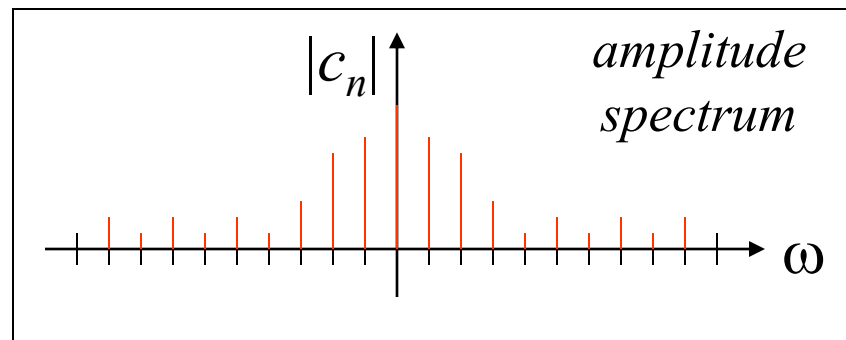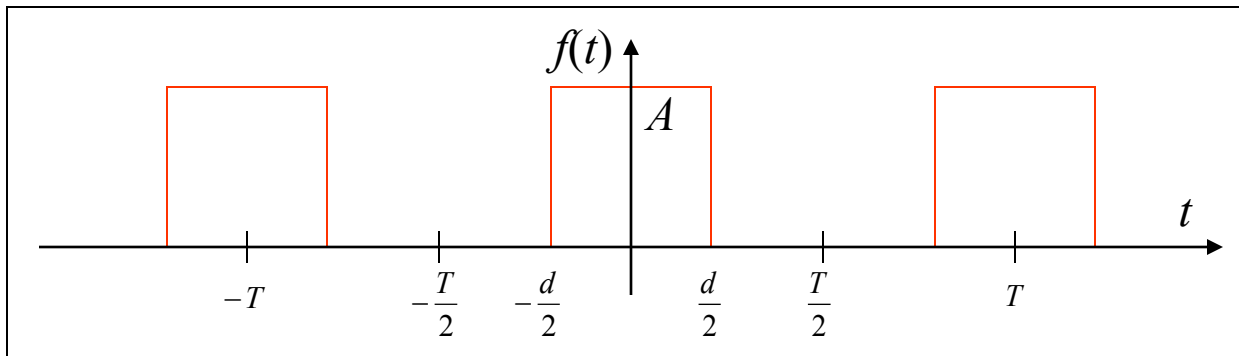
Example

$$c_n = \frac{A}{T} \int_{-d/2}^{d/2} e^{-jn\omega_0 t} dt$$

$$= \frac{A}{T} \frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \Big|_{-d/2}^{d/2}$$

$$= \frac{A}{T}\left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 d/2} - \frac{1}{-jn\omega_0} e^{jn\omega_0 d/2} \right)$$

$$= \frac{A}{T} \frac{1}{-jn\omega_0} (-2j \sin n\omega_0 d/2)$$

$$= \frac{A}{T} \frac{1}{\frac{1}{2} n\omega_0} \sin n\omega_0 d/2$$

$$= \frac{Ad}{T} \frac{\sin\left(\dfrac{n\pi d}{T} \right)}{\left(\dfrac{n\pi d}{T} \right)}$$

Example
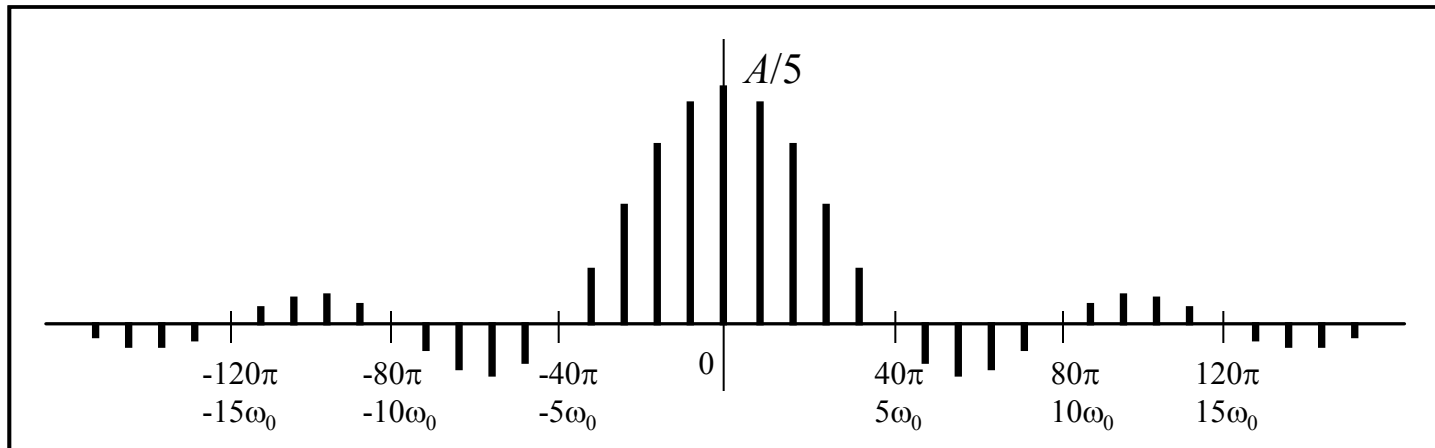
$$c_n = \frac{Ad}{T}\frac{\sin\left(\dfrac{n\pi d}{T}\right)}{\left(\dfrac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20}, \boxed{T = \frac{1}{4},}\ \frac{d}{T} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 8\pi$$

Example
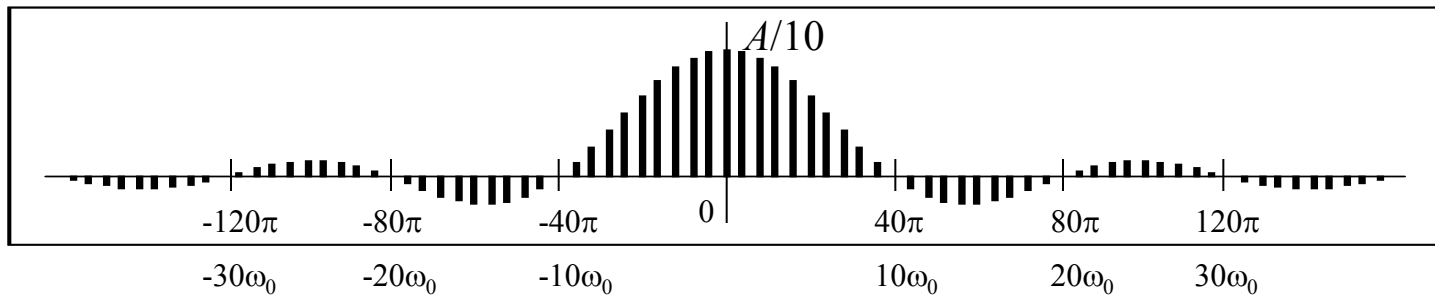
$$c_n = \frac{Ad}{T} \frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20}, \quad \boxed{T = \frac{1}{2}}, \quad \frac{d}{T} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 4\pi$$

Example
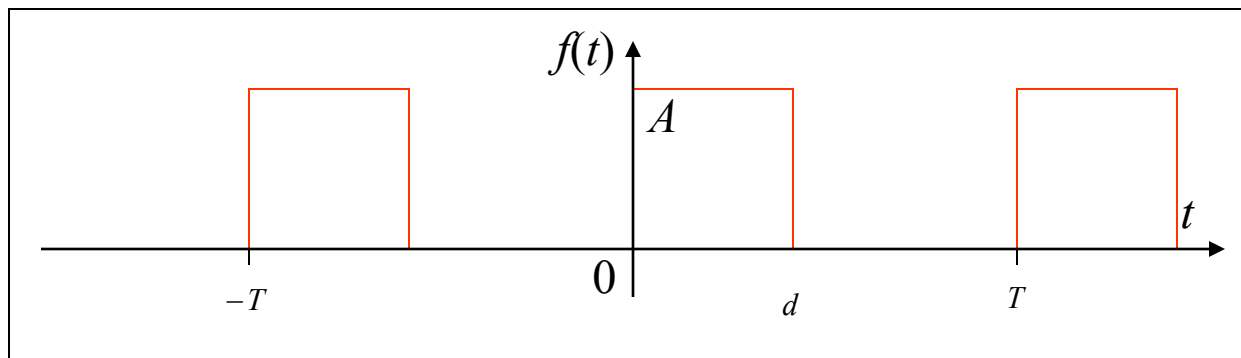
$$c_n = \frac{A}{T} \int_0^d e^{-jn\omega_0 t} dt$$

$$= \frac{A}{T} \frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \Big|_0^d$$

$$= \frac{A}{T} \left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 d} - \frac{1}{-jn\omega_0} \right)$$

$$= \frac{A}{T} \frac{1}{jn\omega_0} (1 - e^{-jn\omega_0 d})$$

$$= \frac{A}{T} \frac{1}{jn\omega_0} e^{-jn\omega_0 d/2} (e^{jn\omega_0 d/2} - e^{-jn\omega_0 d/2})$$

$$= \frac{Ad}{T} \frac{\sin\left(\dfrac{n\pi d}{T}\right)}{\left(\dfrac{n\pi d}{T}\right)} e^{-jn\omega_0 d/2}$$

Fourier Series

Impulse Train

Dirac Delta Function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$
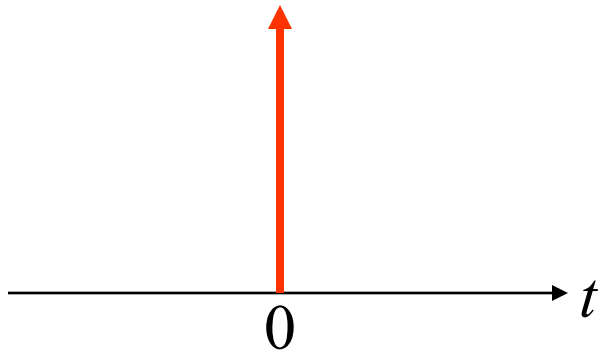
Also called *unit impulse function*.

Property

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0)$$

$\phi(t)$: Test Function

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\phi(0)\,dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)\,dt = \phi(0)$$

Impulse Train
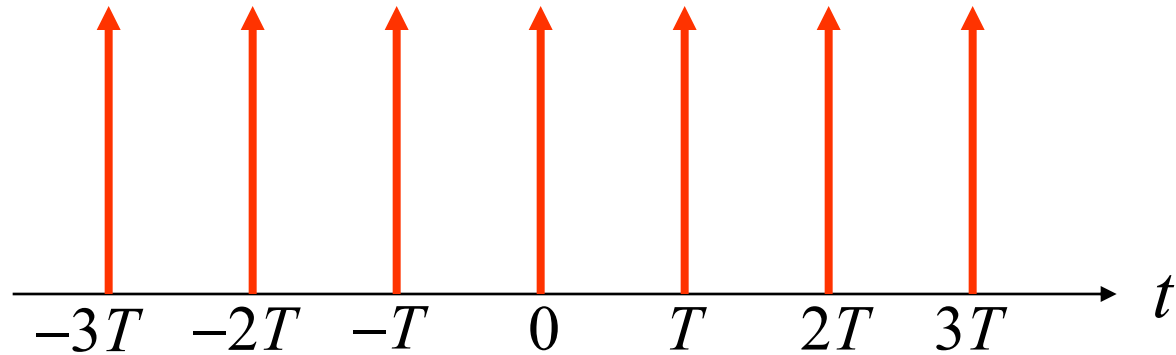
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Fourier Series of the Impulse Train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} \delta_T(t) dt = \frac{2}{T}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} \delta_T(t) \cos(n\omega_0 t) dt = \frac{2}{T}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} \delta_T(t) \sin(n\omega_0 t) dt = 0$$

$$\delta_T(t) = \frac{1}{T} + \frac{2}{T} \sum_{n=-\infty}^{\infty} \cos n\omega_0 t$$

Complex Form
Fourier Series of the Impulse Train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)\,dt = \frac{1}{T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T}$$

$$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Fourier Series

**Analysis of
Periodic Waveforms**

Waveform Symmetry

- Even Functions

$$f(t) = f(-t)$$
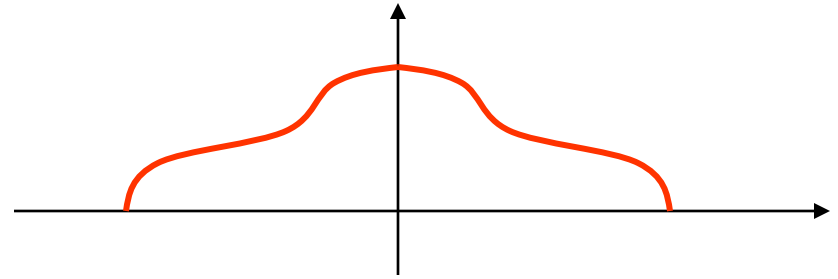
- Odd Functions

$$f(t) = -f(-t)$$
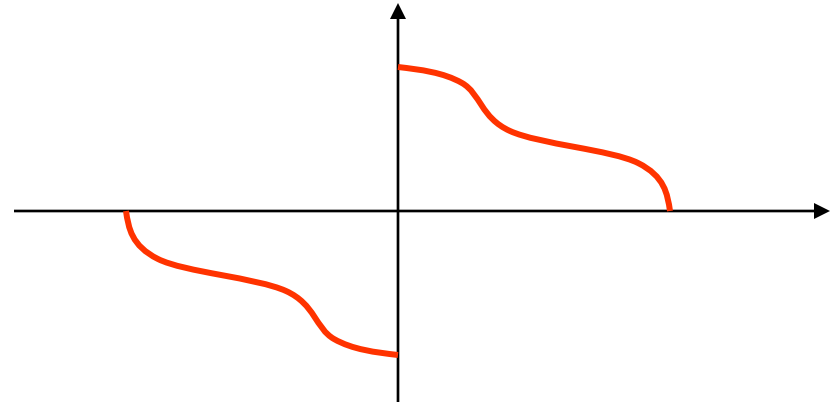
Decomposition

- Any function $f(t)$ can be expressed as the sum of an even function $f_e(t)$ and an odd function $f_o(t)$.

$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \tfrac{1}{2}[f(t) + f(-t)] \qquad \text{Even Part}$$

$$f_o(t) = \tfrac{1}{2}[f(t) - f(-t)] \qquad \text{Odd Part}$$

Example

$$f(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & t < 0 \end{cases}$$
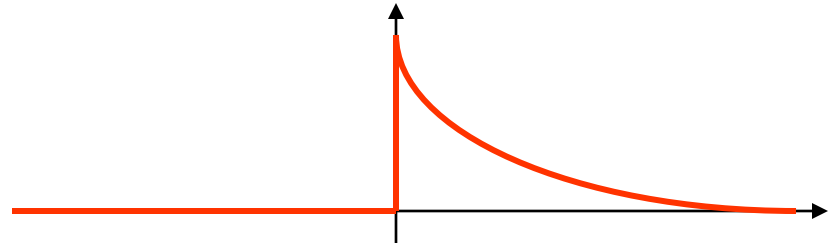
$$f_e(t) = \begin{cases} \frac{1}{2}e^{-t} & t > 0 \\ \frac{1}{2}e^{t} & t < 0 \end{cases}$$

Even Part
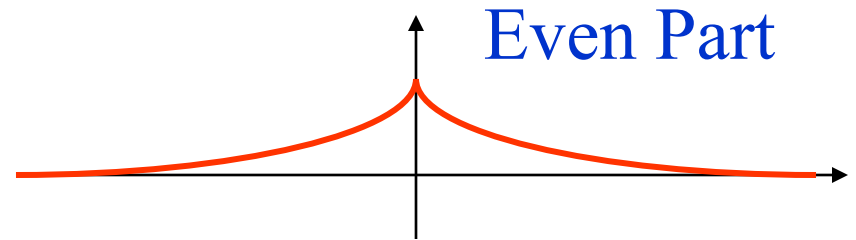
$$f_o(t) = \begin{cases} \frac{1}{2}e^{-t} & t > 0 \\ -\frac{1}{2}e^{t} & t < 0 \end{cases}$$

Odd Part
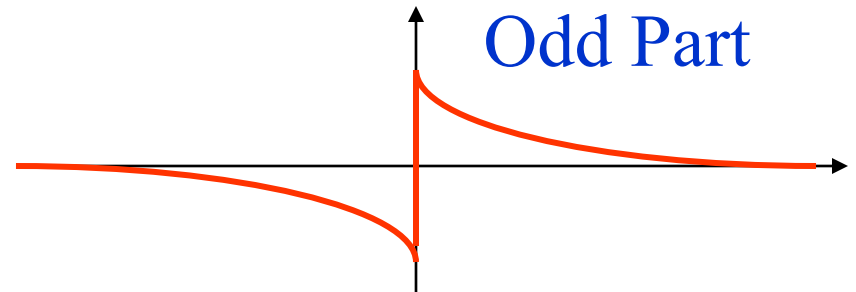
Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f(t+T/2)$$
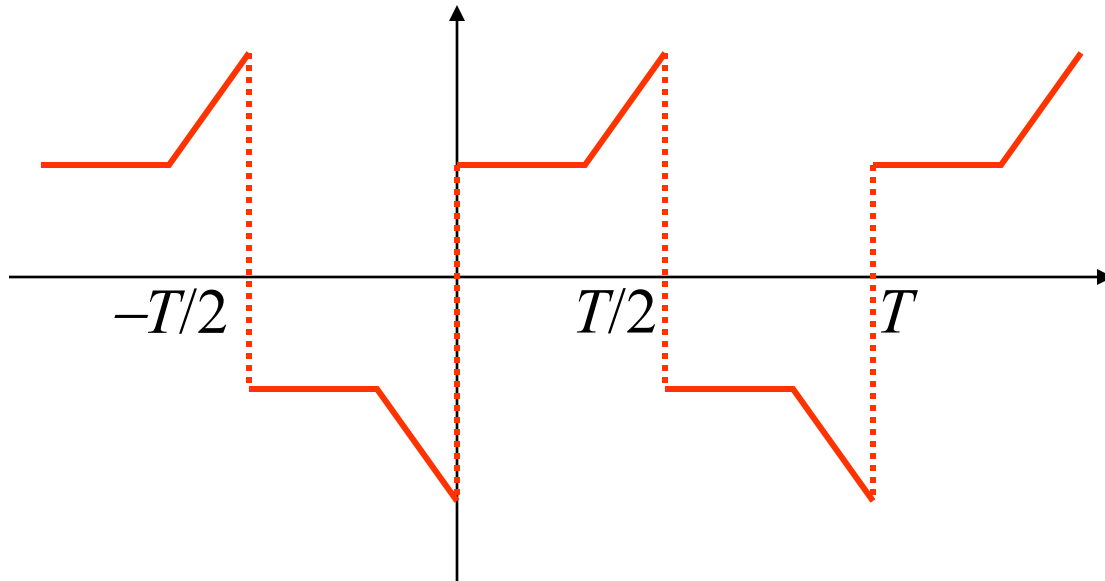
Quarter-Wave Symmetry

Even Quarter-Wave Symmetry
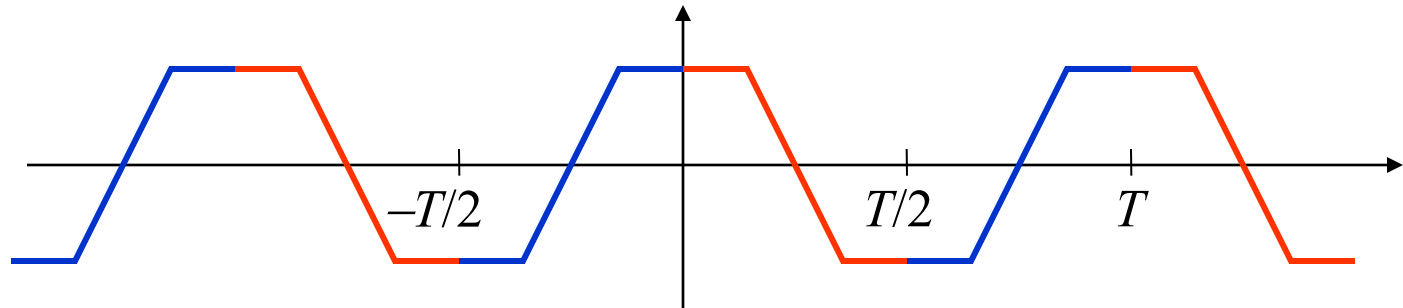
Odd Quarter-Wave Symmetry
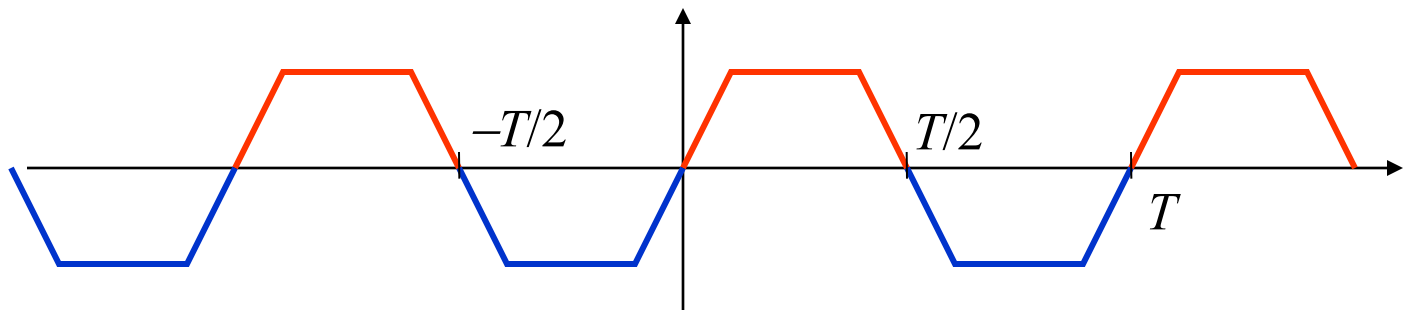
Hidden Symmetry

- The following is a asymmetry periodic function:
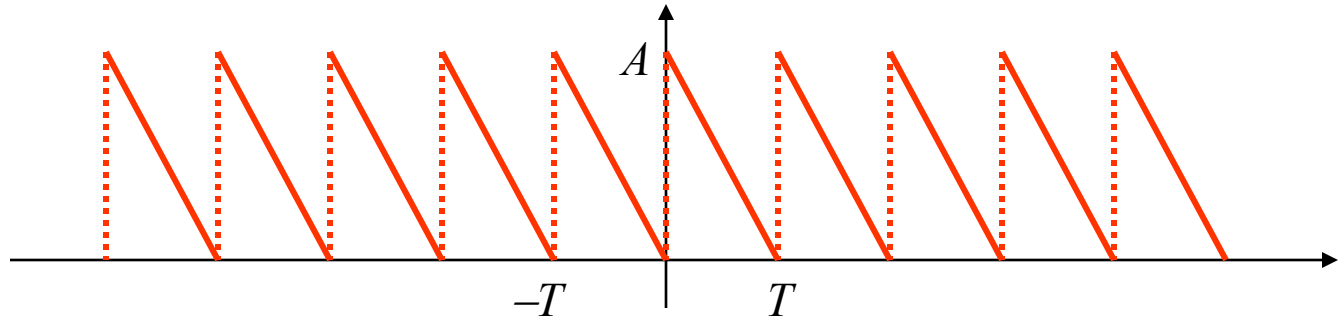
- Adding a constant to get symmetry property.
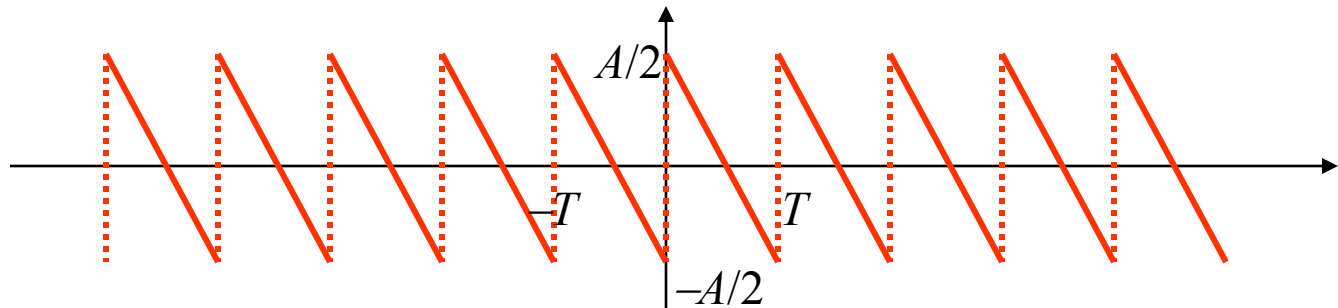
Fourier Coefficients of Symmetrical Waveforms

- The use of symmetry properties simplifies the calculation of Fourier coefficients.
  - Even Functions
  - Odd Functions
  - Half-Wave
  - Even Quarter-Wave
  - Odd Quarter-Wave
  - Hidden

Fourier Coefficients of Even Functions
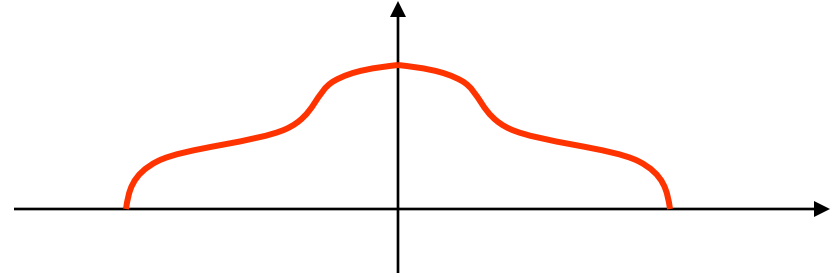
$$f(t) = f(-t)$$
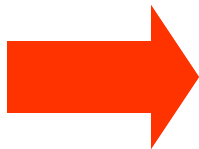

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$
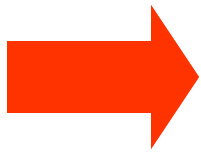
$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(n\omega_0 t) dt$$

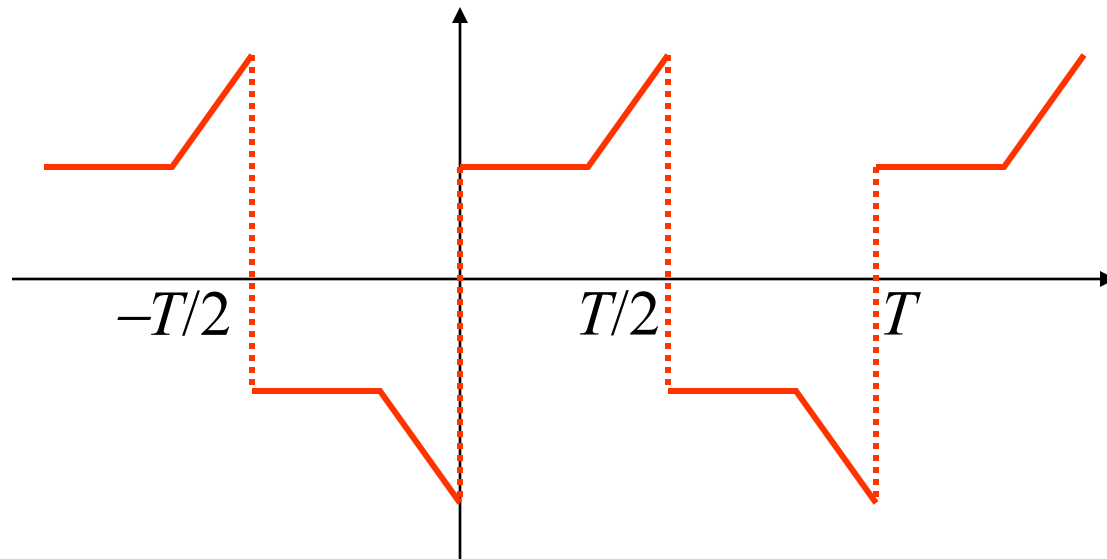# Fourier Coefficients of Even Functions
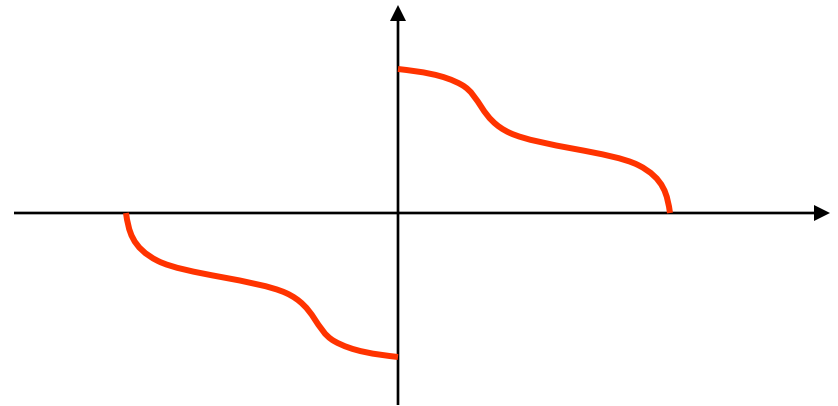
$$f(t) = -f(-t)$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin(n\omega_0 t) dt$$

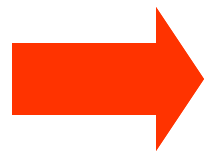# Fourier Coefficients for Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f(t+T/2)$$

The Fourier series contains only odd harmonics.

Fourier Coefficients for Half-Wave Symmetry

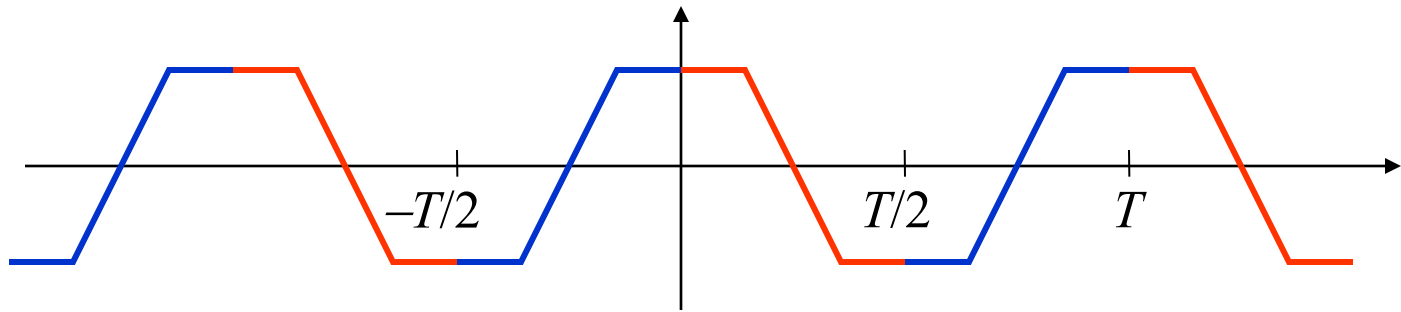$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t + T/2\right)$$

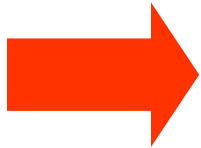$$f(t) = \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt & \text{for } n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt & \text{for } n \text{ odd} \end{cases}$$

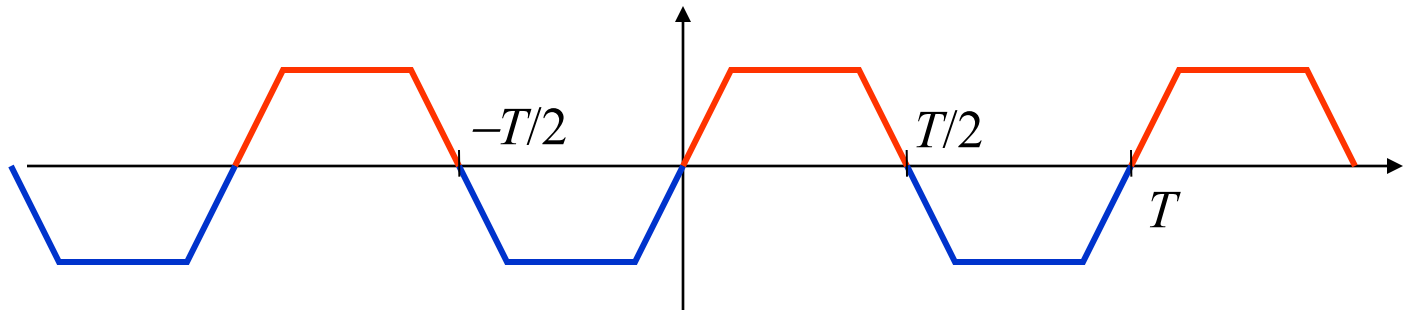# Fourier Coefficients for
Even Quarter-Wave Symmetry

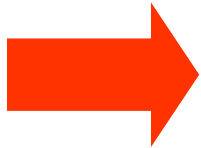$$f(t) = \sum_{n=1}^{\infty} a_{2n-1} \cos[(2n-1)\omega_0 t]$$

$$a_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \cos[(2n-1)\omega_0 t] dt$$

Fourier Coefficients for
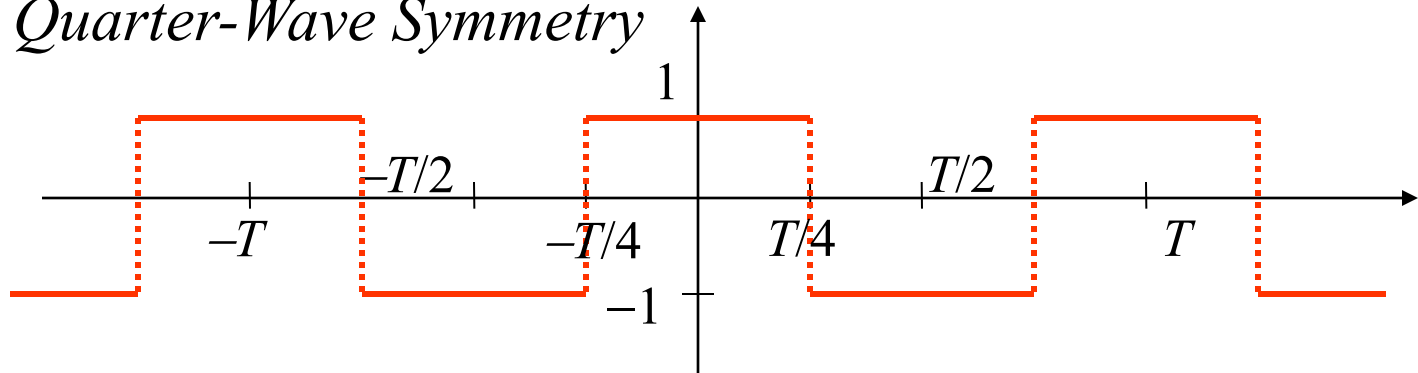# Odd Quarter-Wave Symmetry

$$f(t) = \sum_{n=1}^{\infty} b_{2n-1} \sin[(2n-1)\omega_0 t]$$

$$b_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \sin[(2n-1)\omega_0 t] dt$$
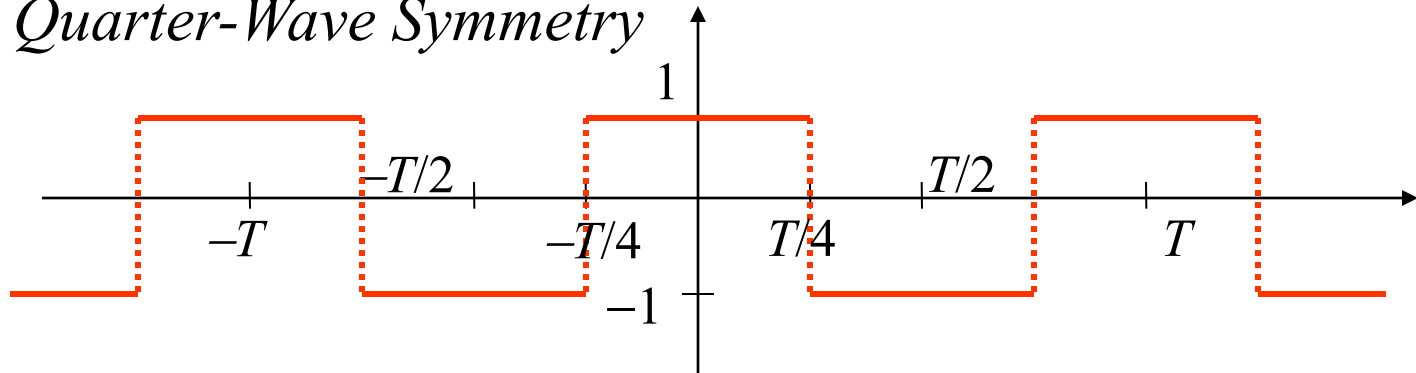
Example

Even Quarter-Wave Symmetry

$$a_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \cos[(2n-1)\omega_0 t] dt = \frac{8}{T} \int_0^{T/4} \cos[(2n-1)\omega_0 t] dt$$

$$= \frac{8}{(2n-1)\omega_0 T} \sin[(2n-1)\omega_0 t] \Big|_0^{T/4} = (-1)^{n-1} \frac{4}{(2n-1)\pi}$$

Example

$$f(t) = \frac{4}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t + \cdots\right)$$

Even Quarter-Wave Symmetry
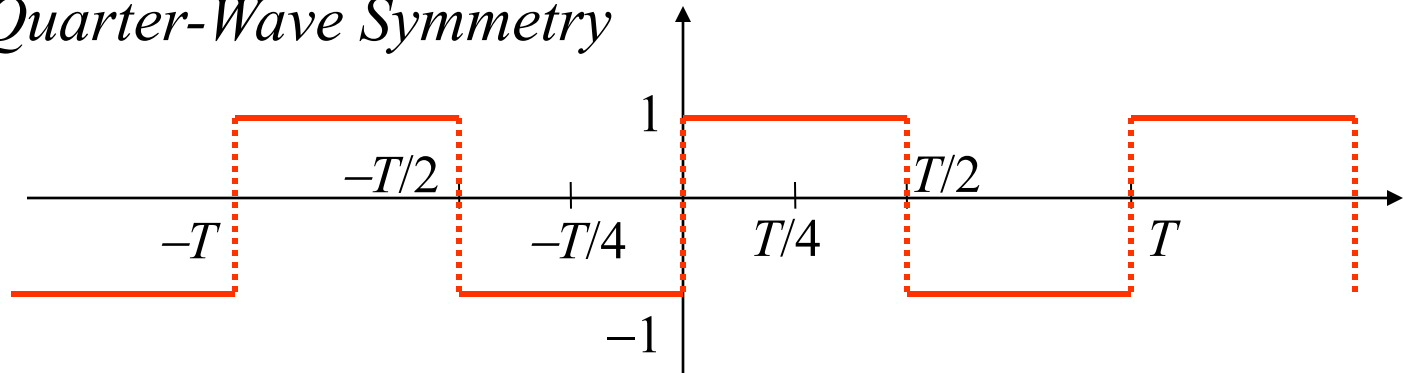

$$a_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\cos[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4}\cos[(2n-1)\omega_0 t]dt$$

$$= \frac{8}{(2n-1)\omega_0 T}\sin[(2n-1)\omega_0 t]\Big|_0^{T/4} = (-1)^{n-1}\frac{4}{(2n-1)\pi}$$
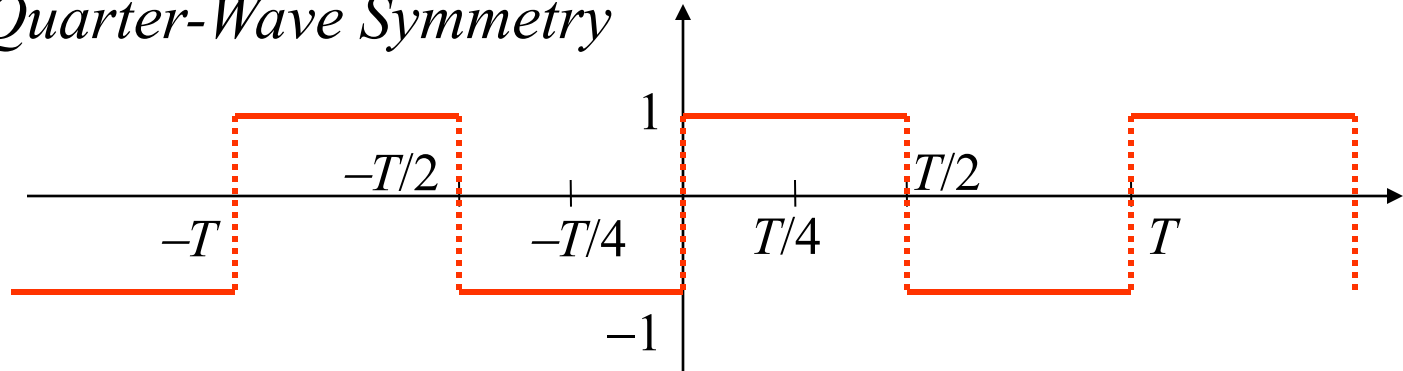
Example

Odd Quarter-Wave Symmetry

$$b_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \sin[(2n-1)\omega_0 t] dt = \frac{8}{T} \int_0^{T/4} \sin[(2n-1)\omega_0 t] dt$$

$$= \frac{-8}{(2n-1)\omega_0 T} \cos[(2n-1)\omega_0 t] \bigg|_0^{T/4} = \frac{4}{(2n-1)\pi}$$

Example

$$f(t) = \frac{4}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$$
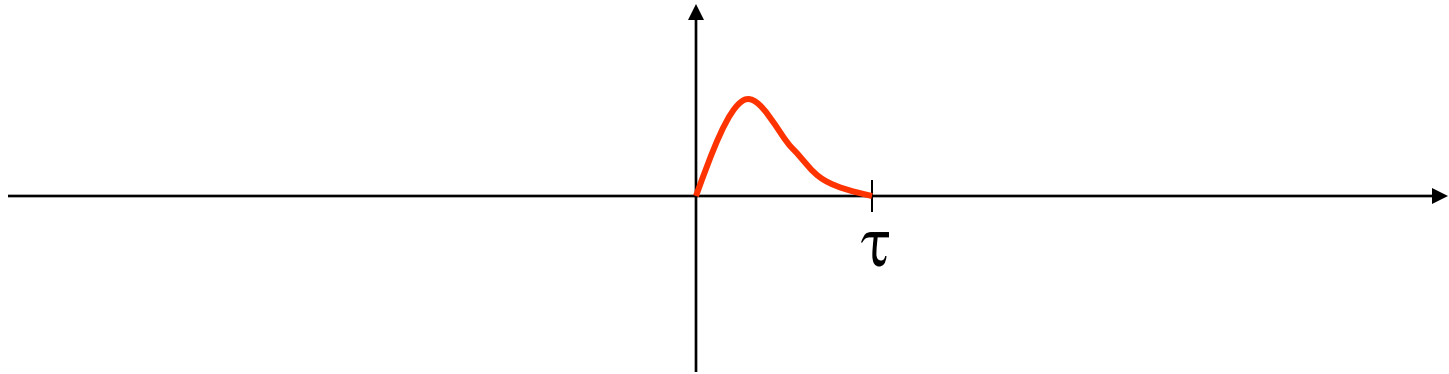
Odd Quarter-Wave Symmetry

$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]\,dt = \frac{8}{T}\int_0^{T/4}\sin[(2n-1)\omega_0 t]\,dt$$

$$= \frac{-8}{(2n-1)\omega_0 T}\cos[(2n-1)\omega_0 t]\Big|_0^{T/4} = \frac{4}{(2n-1)\pi}$$
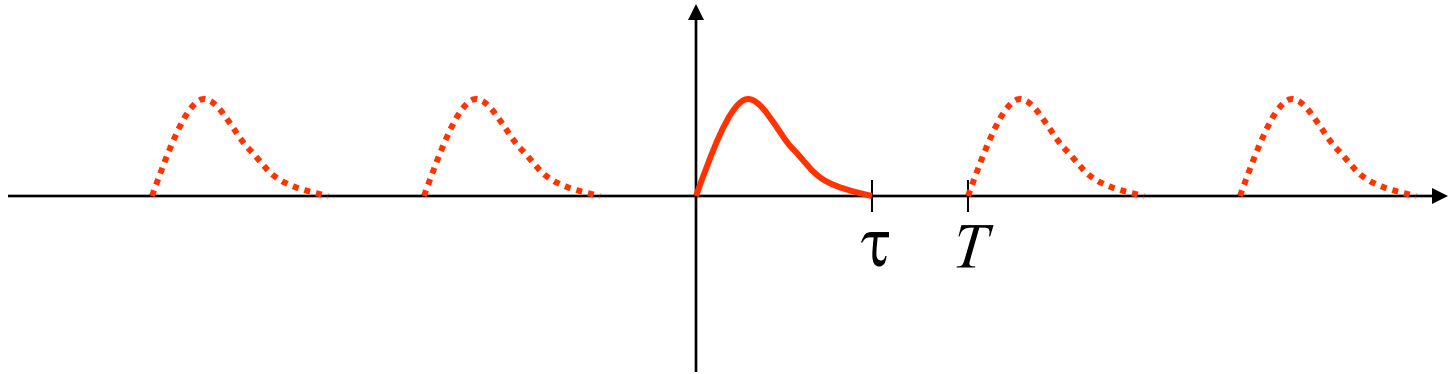
Fourier Series

Half-Range Expansions

Non-Periodic Function Representation

- A non-periodic function $f(t)$ defined over (0, τ) can be expanded into a Fourier series which is defined only in the interval (0, τ).

Without Considering Symmetry

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Symmetry
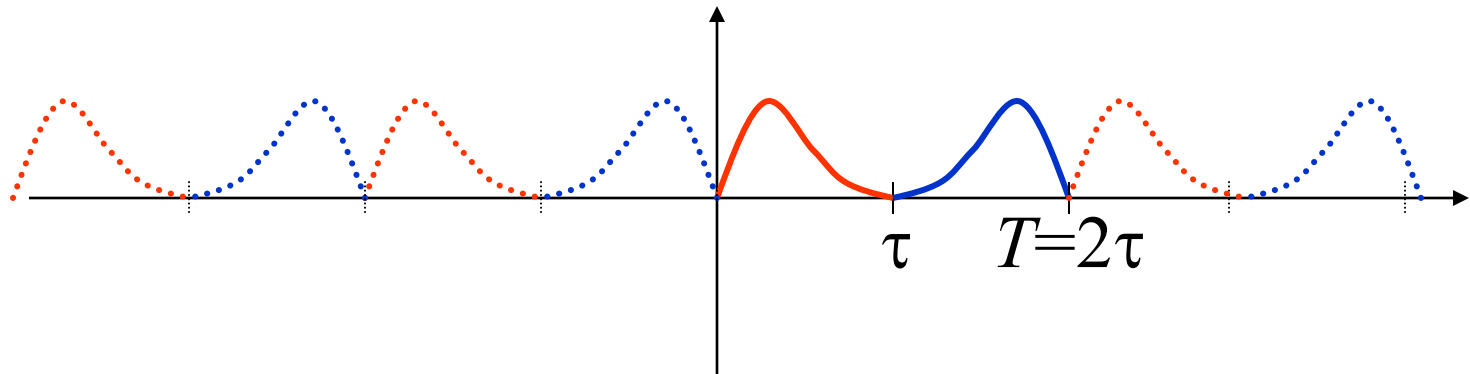

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Odd Symmetry

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Half-Wave Symmetry

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.
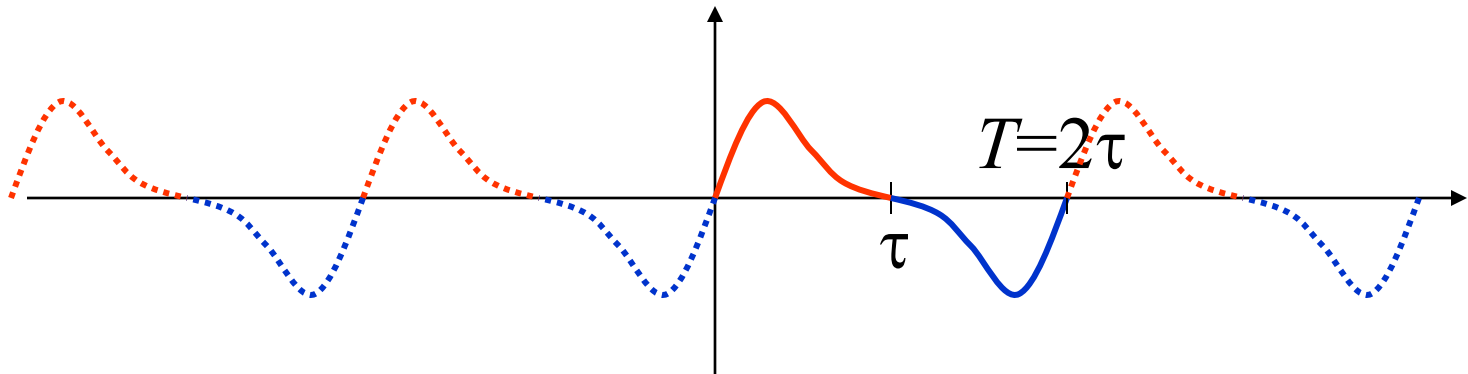
Expansion Into
Even Quarter-Wave Symmetry
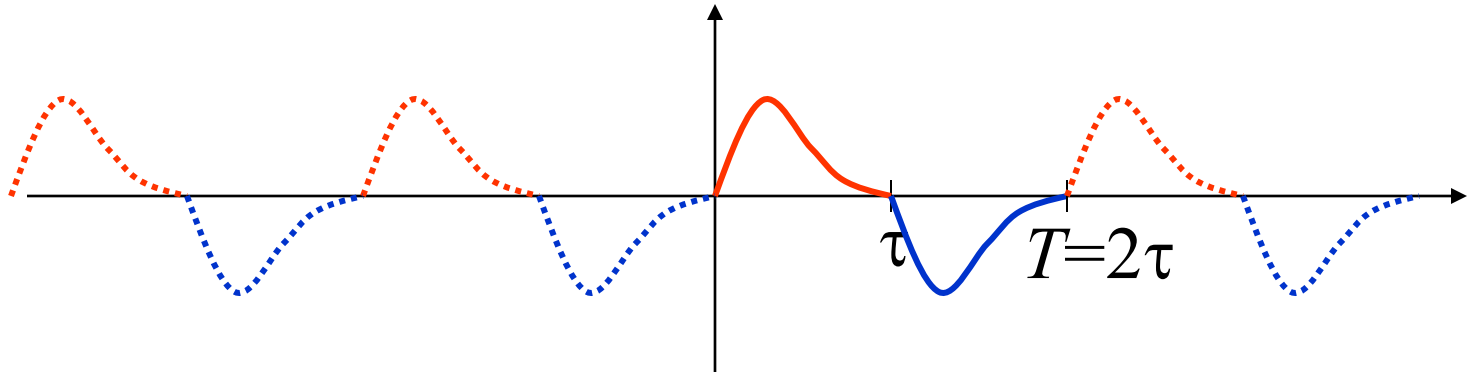

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into
Odd Quarter-Wave Symmetry
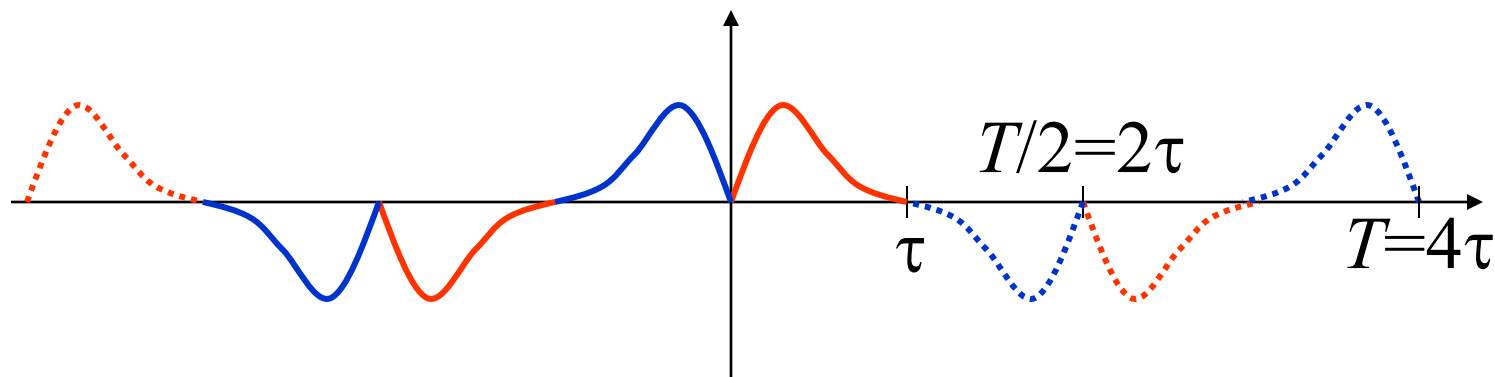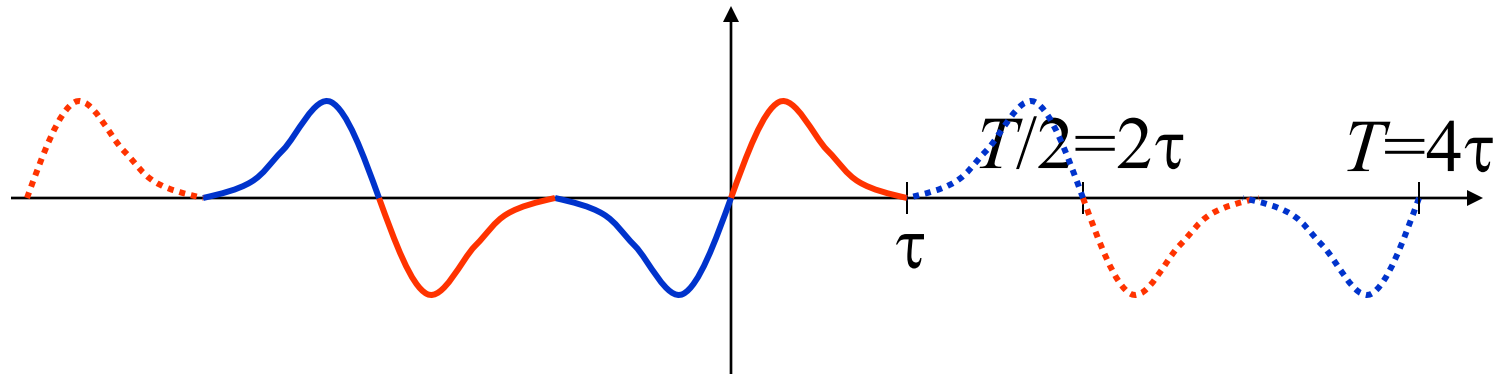

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Fourier Series

Least Mean-Square
Error Approximation

Approximation a function

Use $\quad S_k(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{k}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$

to represent $f(t)$ on interval $-T/2 < t < T/2$.

Define $\varepsilon_k(t) = f(t) - S_k(t)$

$$E_k = \frac{1}{T}\int_{-T/2}^{T/2}[\varepsilon_k(t)]^2\,dt \qquad \text{Mean-Square Error}$$

Approximation a function

Show that using $S_k(t)$ to represent $f(t)$ has least mean-square property.

$$E_k = \frac{1}{T}\int_{-T/2}^{T/2}[\varepsilon_k(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

Proven by setting $\partial E_k/\partial a_i = 0$ and $\partial E_k/\partial b_i = 0$.

Approximation a function

$$E_k = \frac{1}{T} \int_{-T/2}^{T/2} \left[\varepsilon_k(t)\right]^2 dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) \right]^2 dt$$

$$\frac{\partial E_k}{\partial a_0} = \frac{a_0}{2} - \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt = 0$$

$$\frac{\partial E_k}{\partial a_n} = a_n - \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t\, dt = 0$$

$$\frac{\partial E_k}{\partial b_n} = b_n - \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t\, dt = 0$$

Mean-Square Error

$$E_k = \frac{1}{T} \int_{-T/2}^{T/2} [\varepsilon_k(t)]^2 \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right) \right]^2 dt$$

$$E_k = \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt - \frac{a_0^2}{4} - \frac{1}{2} \sum_{n=1}^{k} (a_n^2 + b_n^2)$$

Mean-Square Error

$$E_k = \frac{1}{T}\int_{-T/2}^{T/2}[\varepsilon_k(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt \geq \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{k}(a_n^2 + b_n^2)$$

Mean-Square Error

$$E_k = \frac{1}{T} \int_{-T/2}^{T/2} [\varepsilon_k(t)]^2 \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \right]^2 dt$$

$$\frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$