Method of Functional Separation of Variables

Functional Separation of Variables

General form of exact solutions:

$$w(x,t) = F(z), \quad z = \varphi_1(x)\psi_1(t) + \cdots + \varphi_n(x)\psi_n(t) \qquad (*)$$

In general, the functions $\varphi_m(t)$, $\psi_m(t)$, and $F(z)$ in $(*)$ are not known in advance and are to be identified.

Main idea: the functional-differential equation resulting from the substitution of expression $(*)$ in the original PDE should be reduced to *the standard bilinear functional equation* (Lecture 1: Method of generalized separation of variables).

Functional separable solutions of special form:

$$w = F(z), \quad z = \varphi(t)x + \psi(t)$$

$$w = F(z), \quad z = \varphi(x) + \psi(t)$$

The former solution is called a *generalized travelling-wave solution*.

General Scheme for Constructing Generalized Traveling-Wave Solutions by the Splitting Method for Evolution Equations

Original equation: $w_t = H(t, w, w_x, w_{xx}, \ldots, w_x^{(n)})$

Search for generalized traveling-wave solutions

Define solution structure: $w = F(z)$, where $z = \varphi(t)\,x + \psi(t)$

Substitute into original equation and replace x by $(z - \psi)/\varphi$

Write out the functional-differential equation in two arguments

Apply splitting procedure

Obtain: (i) functional equation, (ii) determining system of ODEs

Treat functional equation (i)

Solve the functional equation:

Solve the determining system of ordinary differential equations

Find the functions φ, ψ and F

Write out generalized traveling-wave solution of original equation

Example 1. Nonlinear Heat Equation

Consider the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(w) \qquad (1)$$

We look for generalized traveling-wave solutions of the form

$$(2)$$

The functions $w(z)$, $\varphi(t)$, $\psi(t)$, and $f(w)$ are to be determined. Substitute (2) into (1) and divide by w'_z to obtain

$$\varphi'_t x + \psi'_t = \varphi^2 \frac{w''_{zz}}{w'_z} + \frac{f(w)}{w'_z} \qquad (3)$$

On expressing x from (2) in terms of z and substituting into (3), we get a functional-differential equation in two variables, t and z,

Example 1. Nonlinear Heat Equation (continued)

The functional differential equation

can be rewritten as the standard bilinear functional equation

with

Substituting these expressions into the solution of the 4-term functional equation (Lecture 1) yields the determining system of ordinary differential equations

where A_1, A_2, A_3, A_4 are arbitrary constants.

Example 1. Nonlinear Heat Equation (continued)

Determining system of ordinary differential equations:

The solution to the determining system of ODEs is given by

where $C_1, \ldots, C_4$ are arbitrary parameters, $A_4 \neq 0$. The dependence $f = f(w)$ is defined by the last two relations in parametric form (z is treated as the parameter).

Example 2. Nonlinear Heat Equation

Again consider the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(w) \qquad (1)$$

We now look for functional separable solutions of the special form

$$w = w(z), \quad z = \varphi(x) + \psi(t) \qquad (2)$$

Substitute (2) into (1) and divide by w'_z to obtain

$$\psi'_t = \varphi''_{xx} + (\varphi'_x)^2 Q(z) + R(z) \qquad (3)$$

where

$$Q(z) = \frac{w''_{zz}}{w'_z}, \quad R(z) = \frac{f(w)}{w'_z}$$

Differentiating (3) with respect to z yields

where

$$\begin{gathered} \Phi_1 = \varphi'''_{xxx}, \quad \Phi_2 = \varphi'_x \varphi''_{xx}, \\ \Phi_3 = (\varphi'_x)^3, \quad \Phi_4 = \varphi'_x, \\ \Psi_1 = 1, \quad \Psi_2 = 2Q(z), \\ \Psi_3 = Q'_z(z), \quad \Psi_4 = R'_z(z) \end{gathered} \qquad (5)$$

Expressions (5) should then be substituted into the solution of the functional equation (4) to obtain the determining system of ODEs (see Lecture 1).

Example 3. Mass and Heat Transfer with Volume Reaction

Nonlinear equation:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x} \right] + g(w)$$

First functional separable solution

Let the function $f = f(w)$ be arbitrary and let $g = g(w)$ be defined by

$$g(w) = \frac{A}{f(w)} + B, \quad A, B \text{ are arbitrary numbers}$$

In this case, there is a functional separable solution defined implicitly by

$$\int f(w)\,dw = At - \tfrac{1}{2}Bx^2 + C_1 x + C_2$$

where C_1, C_2 are arbitrary numbers.

Mass and Heat Transfer with Volume Reaction (continued)

Nonlinear equation:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w) \frac{\partial w}{\partial x} \right] + g(w)$$

Second functional separable solution

Let now $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$(*)$

where A_1, A_2, A_3 are some numbers. Then there are generalized traveling-wave solutions of the form

where $w(Z)$ is determined by inverting the second relation in $(*)$ and C_1, C_2 are arbitrary numbers.

Mass and Heat Transfer with Volume Reaction (continued)

Nonlinear equation:

Third functional separable solution

Let now $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$(*)$

where $A_4 \neq 0$. Then there are generalized traveling-wave solutions of the form

$$w = w(Z), \quad Z = \varphi(t)\,x + \psi(t)$$

where the function $w(Z)$ is determined by the inversion of relation $(*)$

and C_1, C_2 are arbitrary numbers.

Mass and Heat Transfer with Volume Reaction (continued)

Nonlinear equation:

Fourth functional separable solution

Let the functions $f = f(w)$ and $g = g(w)$ be defined as follows:

where $\varphi(w)$ is an arbitrary function and a, b, c are any numbers; the prime denotes the derivative with respect to w. Then there are functional separable solutions defined implicitly by

Mass and Heat Transfer with Volume Reaction (continued)

Nonlinear equation:

Fifth functional separable solution

Let the functions $f = f(w)$ and $g = g(w)$ be defined as follows:

where $\varphi(w)$ is an arbitrary function and a is any numbers; the prime denotes the derivative with respect to w. Then there are functional separable solutions defined implicitly by

where C_1, C_2 are arbitrary numbers.

Mass and Heat Transfer with Volume Reaction (continued)

Nonlinear equation:

Sixth functional separable solution

Let the functions $f = f(w)$ and $g = g(w)$ be defined as follows:

where $V(z)$ is an arbitrary function of z; A, B are arbitrary constants ($AB \neq 0$); and the function $z = z(w)$ is determined implicitly by

$$w = \int z^{-1/2} V_z'(z)\, dz + C_1 \qquad (*)$$

with C_1 being an arbitrary constant. Then there is a functional separable solution of the form ($*$) where

with C_2, C_3 being arbitrary constants.

Nonlinear Schrödinger Equation with Cubic Nonlinearity

Equation:

Exact solution:

where the functions $a = a(t),\ b = b(t),\ \alpha = \alpha(t),\ \beta = \beta(t),\ \gamma = \gamma(t)$ are determined by the system of ODEs

$$
\begin{aligned}
a'_t &= -6a\alpha \\
b'_t &= -2a\beta - 2b\alpha \\
\alpha'_t &= f(t)\,a^2 - 4\alpha^2 \\
\beta'_t &= 2f(t)\,ab - 4\alpha\beta \\
\gamma'_t &= f(t)\,b^2 - \beta^2 + g(t)
\end{aligned}
$$

Nonlinear Schrödinger Equation of General Form

Equation:

$$i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+f(|w|)\,w=0$$

1. Exact solution:

where A, B, C are arbitrary real constants, and the function $\varphi=\varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{zz}+f(|\varphi|)\varphi-(Az+B)\varphi=0$$

2. Exact solutions:

where C_1, C_2, C_3 are arbitrary real constants.