Partial Differential Equations

- Definition
- One of the classical partial differential equation of mathematical physics is the equation describing the conduction of heat in a solid body (Originated in the 18th century). And a modern one is the space vehicle reentry problem: Analysis of transfer and dissipation of heat generated by the friction with earth's atmosphere.

For example:

- Consider a straight bar with uniform cross-section and homogeneous material. We wish to develop a model for heat flow through the bar.
- Let $u(x,t)$ be the temperature on a cross section located at x and at time t. We shall follow some basic principles of physics:
- A. The amount of heat per unit time flowing through a unit of cross-sectional area is proportional to $\partial u / \partial x$ with constant of proportionality $k(x)$ called the thermal conductivity of the material.

- B. Heat flow is always from points of higher temperature to points of lower temperature.
- C. The amount of heat necessary to raise the temperature of an object of mass "m" by an amount Δu is a "c(x) m Δu", where c(x) is known as the specific heat capacity of the material.
- Thus to study the amount of heat H(x) flowing from left to right through a surface A of a cross section during the time interval Δt can then be given by the formula:

$$H(x) = -k(x)(\textit{area} \text{ of A})\Delta t \frac{\partial u}{\partial x}(x,t)$$

Likewise, at the point $x + \Delta x$, we have

- Heat flowing from left to right across the plane during an time interval Δt is:

$$H(x+\Delta x) = -k(x+\Delta x)(area\text{ of B})\Delta \text{t}\frac{\partial \text{u}}{\partial t}(x+\Delta x, t).$$

- If on the interval $[x, x+\Delta x]$, during time Δt , additional heat sources were generated by, say, chemical reactions, heater, or electric currents, with energy density $Q(x,t)$, then the total change in the heat ΔE is given by the formula:

ΔE = Heat entering A - Heat leaving B + Heat generated .

- And taking into simplification the principle C above, $\Delta E = c(x)\, m\, \Delta u$, where $m = \rho(x)\, \Delta V$. After dividing by $(\Delta x)(\Delta t)$, and taking the limits as Δx , and $\Delta t \rightarrow 0$, we get:

$$\frac{\partial}{\partial x}\left[k(x)\frac{\partial u}{\partial x}(x,t)\right]+Q(x,t)=c(x)\rho(x)\frac{\partial u}{\partial t}(x,t)$$

- If we assume k, c, ρ are constants, then the eq. Becomes:

$$\frac{\partial u}{\partial t}=\beta^{2}\frac{\partial^{2}u}{\partial x^{2}}+p(x,t)$$

Boundary and Initial conditions

- Remark on boundary conditions and initial condition on $u(x,t)$.
- We thus obtain the mathematical model for the heat flow in a uniform rod without internal sources ($p = 0$) with homogeneous boundary conditions and initial temperature distribution $f(x)$, the follolwing Initial Boundary Value Problem:

One Dimensional Heat Equation

$$\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial x^2}(x,t), 0 < x < L,\ t > 0,$$

$$u(0,t) = u(L,t) = 0, \quad t > 0,$$

$$u(x,0) = f(x), \quad 0 < x < L.$$

The method of separation of variables

- Introducing solution of the form
- u(x,t) = X(x) T(t) .
- Substituting into the I.V.P, we obtain:

$$X(x)T'(t) = \beta X''(x)T(t), \quad 0 < x < L, \quad t > 0.$$

this leads to the following eq.

$$\frac{T'(t)}{\beta T(t)} = \frac{X''(x)}{X(x)} = \text{Constants. Thus we have}$$

$$T'(t) - \beta kT(t) = 0 \text{ and } X''(x) - kX(x) = 0.$$

Boundary Conditions

- Imply that we are looking for a non-trivial solution X(x), satisfying:

$$X''(x) - kX(x) = 0$$

$$X(0) = X(L) = 0$$

- We shall consider 3 cases:
- $k = 0$, $k > 0$ and $k < 0$.

- Case (i): $k = 0$. In this case we have
- $X(x) = 0$, trivial solution
- Case (ii): $k > 0$. Let $k = \lambda^2$, then the D.E gives $X'' - \lambda^2 X = 0$. The fundamental solution set is: $\{ e^{\lambda x}, e^{-\lambda x} \}$. A general solution is given by: $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$
- $X(0) = 0 \Rightarrow c_1 + c_2 = 0$, and
- $X(L) = 0 \Rightarrow c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0$, hence
- $c_1 (e^{2\lambda L} - 1) = 0 \Rightarrow c_1 = 0$ and so is $c_2 = 0$.
- Again we have trivial solution $X(x) \equiv 0$.

Finally Case (iii) when $k < 0$.

- We again let $k = -\lambda^2$, $\lambda > 0$. The D.E. becomes:
- $X''(x) + \lambda^2 X(x) = 0$, the auxiliary equation is
- $r^2 + \lambda^2 = 0$, or $r = \pm \lambda i$. The general solution:
- $X(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$ or we prefer to write:
- $X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$. Now the boundary conditions $X(0) = X(L) = 0$ imply:
- $c_1 = 0$ and $c_2 \sin \lambda L = 0$, for this to happen, we need $\lambda L = n\pi$, i.e. $\lambda = n\pi / L$ or $k = -(n\pi / L)^2$.
- We set $X_n(x) = a_n \sin (n\pi / L)x$, $n = 1, 2, 3, \ldots$

Finally for $T'(t) - \beta k T(t) = 0,\ k = -\lambda^2$.

- We rewrite it as: $T' + \beta\lambda^2 T = 0$. Or $T' = -\beta\lambda^2 T$. We see the solutions are

$$T_n(t) = b_n e^{-\beta (n\pi/L)^2 t}, \quad n = 1, 2, 3, \ldots$$

Thus the function

$u_n(x,t) = X_n(x)T_n(t)$ satisfies the D.E and the boundary conditions. To satisfy the initial condition, we try :

$u(x,t) = \sum u_n(x,t)$, over all n.

- More precisely,

$$u(x,t) = \sum_{1}^{\infty} c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}\right)x \, .$$

We must have:

$$u(x,0) = \sum_{1}^{\infty} c_n \sin\left(\frac{n\pi}{L}\right)x = f(x) \, .$$

- This leads to the question of when it is possible to represent f(x) by the so called
- Fourier sine series ??

Jean Baptiste Joseph Fourier (1768 - 1830)

- Developed the equation for heat transmission and obtained solution under various boundary conditions (1800 - 1811).
- Under Napoleon he went to Egypt as a soldier and worked with G. Monge as a cultural attache for the French army.

Example

- Solve the following heat flow problem

$$\frac{\partial u}{\partial t} = 7\frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < x < \pi \quad , \quad t > 0.$$

$$u(0,t) = u(\pi\ ,t) \ = \ 0\ , \quad t \ > \ 0\ ,$$

$$u(x,0) = 3\sin 2x - 6\sin 5x \ , \ 0 \ < \ x \ < \ \pi.$$

- Write 3 sin 2x - 6 sin 5x = $\sum c_n \sin(n\pi/L)x$, and comparing the coefficients, we see that $c_2 = 3$, $c_5 = -6$, and $c_n = 0$ for all other n. And we have $u(x,t) = u_2(x,t) + u_5(x,t)$.

Wave Equation

- In the study of vibrating string such as piano wire or guitar string.

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}\ , \qquad 0 < x < L\ , \quad t > 0\ ,$$

$$u(0,t) = u(L,t), \quad t > 0\ ,$$

$$u(x,0) = f(x)\ , \quad 0 < x < L\ ,$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \ \ 0 < x < L\ .$$

Example:

- f(x) = 6 sin 2x + 9 sin 7x - sin 10x , and
- g(x) = 11 sin 9x - 14 sin 15x.
- The solution is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi\alpha}{L} t + b_n \sin \frac{n\pi\alpha}{L} t] \sin \frac{n\pi x}{L}.$$

Reminder:

- TA's Review session
- Date: July 17 (Tuesday, for all students)
- Time: 10 - 11:40 am
- Room: 304 BH

Final Exam

- Date: July 19 (Thursday)
- Time: 10:30 - 12:30 pm
- Room: LC-C3
- Covers: all materials
- I will have a review session on Wednesday

Fourier Series

- For a piecewise continuous function f on [-T,T], we have the Fourier series for f:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos\left(\frac{n\pi}{T}\right)x + b_n \sin\left(\frac{n\pi}{T}\right)x\},$$

where

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx, \text{ and}$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(\frac{n\pi}{T}\right)x\,dx;\ \text{n} = 1,2,3,\cdots$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\left(\frac{n\pi}{T}\right)x\,dx;\ \text{n} = 1,2,3,\cdots$$

Examples

- Compute the Fourier series for

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

$$g(x) = |x|, \quad -1 < x < 1.$$

Convergence of Fourier Series

- Pointwise Convegence
- Theorem. If f and f ′ are piecewise continuous on [-T, T], then for any x in (-T, T), we have

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{T}+b_n\sin\frac{n\pi x}{T}\right\}=\frac{1}{2}\left\{f(x^+)+f(x^-\right\},$$

- where the a_n's and b_n's are given by the previous fomulas. It converges to the average value of the left and right hand limits of f(x). Remark on x = T, or -T.

Fourier Sine and Cosine series

- Consider Even and Odd extensions;
- Definition: Let f(x) be piecewise continuous on the interval [0,T]. The Fourier cosine series of f(x) on [0,T] is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{T}\right)x, \quad a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{n\pi}{T}\right)x\,dx$$

- and the Fourier sine series is:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{T}\right)x, \quad b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{n\pi}{T}\right)x\,dx,$$

$$n = 1,2,3,\cdots$$

Consider the heat flow problem:

$$(1) \qquad \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi \qquad t > 0,$$

$$(2) \qquad u(0, t) = u(\pi, \mathrm{t}), \qquad t > 0,$$

$$(3) \qquad u(x, 0) = \begin{cases} x, & 0 < x \le \dfrac{\pi}{2}, \\ \pi - x, & \dfrac{\pi}{2} \le x < \pi \end{cases}$$

Solution

- Since the boundary condition forces us to consider sine waves, we shall expand f(x) into its Fourier Sine Series with $T = \pi$. Thus

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

With the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-2n^2 t} \sin nx$$

where

$$b_n = \begin{cases} 0, & \text{if n is even,} \\ \dfrac{4(-1)^{(n-1)/2}}{n^2 \pi} & \text{when n is odd.} \end{cases}$$