Mechanics of Rigid Bodies

Rigid body

- **Rigid body**: a system of mass points subject to the holonomic constraints that the distances between all pairs of points remain constant throughout the motion

- If there are N free particles, there are 3N degrees of freedom

- For a rigid body, the number of degrees of freedom is reduced by the constraints expressed in the form:

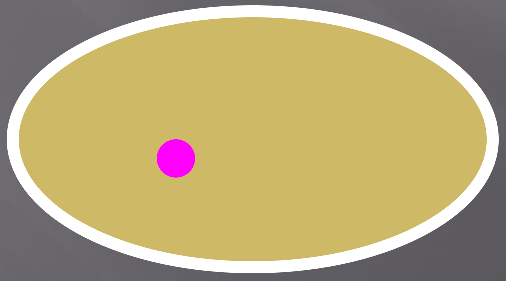
$$r_{ij} = c_{ij}$$

- How many independent coordinates does a rigid body have?

The independent coordinates of a rigid body

- Rigid body has to be described by its orientation and location

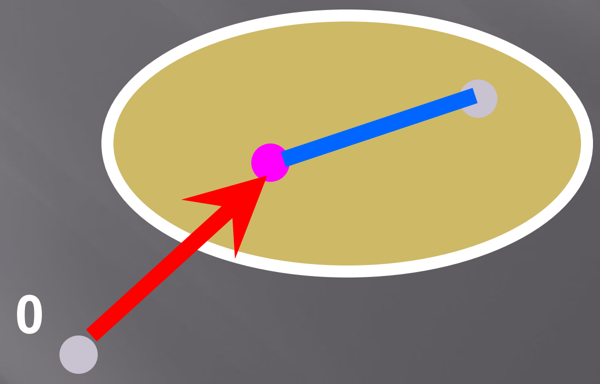
- Position of the rigid body is determined by the position of any one point of the body, and the orientation is determined by the relative position of all other points of the body relative to that point



The independent coordinates of a rigid body

- Position of one point of the body requires the specification of 3 independent coordinates

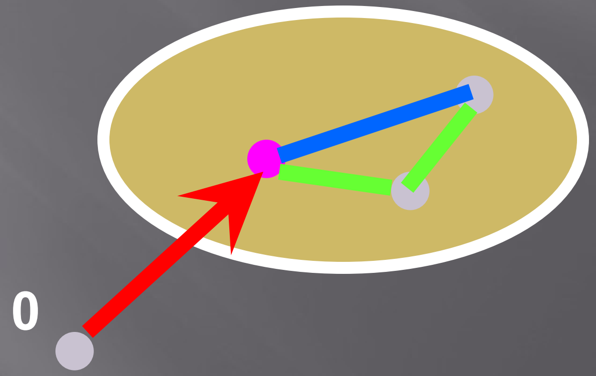
- The position of a second point lies at a fixed distance from the first point, so it can be specified by 2 independent angular coordinates



The independent coordinates of a rigid body

- The position of any other third point is determined by only 1 coordinate, since its distance from the first and second points is fixed

- Thus, the total number of independent coordinates necessary do completely describe the position and orientation of a rigid body is 6



Orientation of a rigid body

- The **position** of a rigid body can be described by **three** independent coordinates,

- Therefore, the **orientation** of a rigid body can be described by the **remaining three** independent coordinates

- There are many ways to define the three orientation coordinates

- One common ways is via the definition of direction cosines

Direction cosines

- **Direction cosines** specify the orientation of one Cartesian set of axes relative to another set with common origin

$$\hat{i}' = \hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13}$$

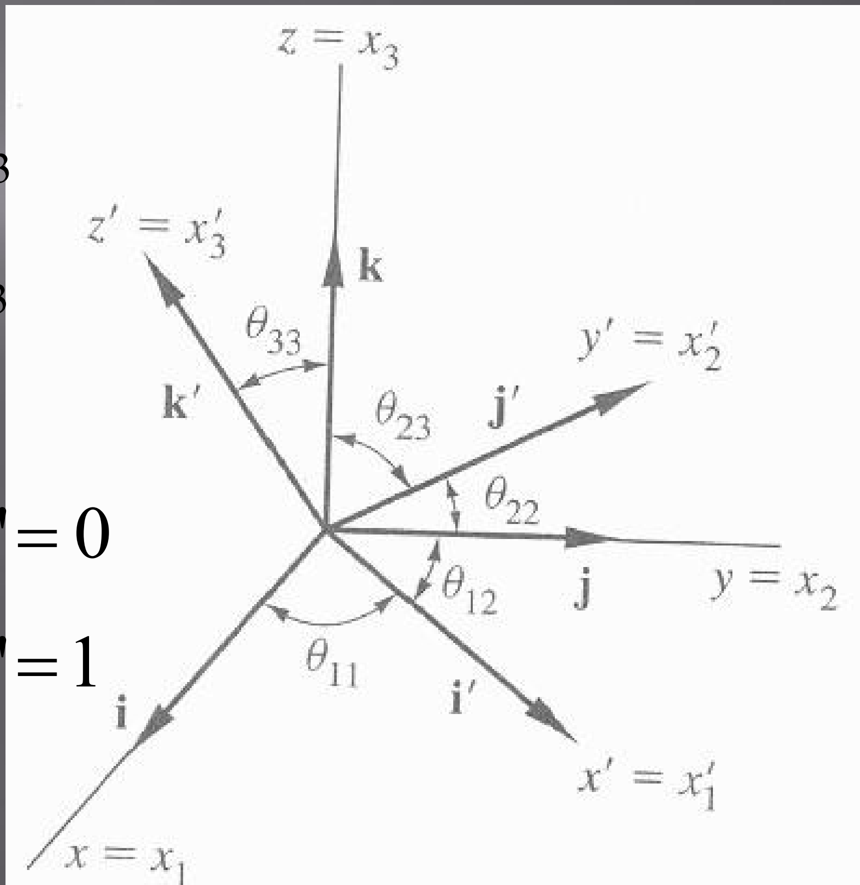
$$\hat{j}' = \hat{i}\cos\theta_{21} + \hat{j}\cos\theta_{22} + \hat{k}\cos\theta_{23}$$

$$\hat{k}' = \hat{i}\cos\theta_{31} + \hat{j}\cos\theta_{32} + \hat{k}\cos\theta_{33}$$

- **Orthogonality conditions**:

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{i}'\cdot\hat{j}' = \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = 0$$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = \hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1$$



Orthogonality conditions

$$\hat{i}'\cdot\hat{i}' =$$

$$= (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot(\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})$$

$$= \cos^2\theta_{11} + \cos^2\theta_{12} + \cos^2\theta_{13} = 1$$

$$\hat{i}'\cdot\hat{j}' =$$

$$= (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot(\hat{i}\cos\theta_{21} + \hat{j}\cos\theta_{22} + \hat{k}\cos\theta_{23})$$

$$= \cos\theta_{11}\cos\theta_{21} + \cos\theta_{12}\cos\theta_{22} + \cos\theta_{13}\cos\theta_{23} = 0$$

- **Performing similar operations for the remaining 4 pairs we obtain orthogonality conditions in a compact form:**

$$\sum_{l=1}^{3}\cos\theta_{li}\cos\theta_{lk} = \delta_{ik}$$

Orthogonal transformations

- For an arbitrary vector $\vec{G} = \hat{i}G_1 + \hat{j}G_2 + \hat{k}G_3$

- We can find components in the primed set of axes as follows: $G_1' = \hat{i}'\cdot\vec{G} = \hat{i}'\cdot\hat{i}G_1 + \hat{i}'\cdot\hat{j}G_2 + \hat{i}'\cdot\hat{k}G_3$

$$= (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot\hat{i}G_1$$

$$+ (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot\hat{j}G_2$$

$$+ (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot\hat{k}G_3$$

$$= \cos\theta_{11}G_1 + \cos\theta_{12}G_2 + \cos\theta_{13}G_3$$

- Similarly

$$G_2' = \cos\theta_{21}G_1 + \cos\theta_{22}G_2 + \cos\theta_{23}G_3$$

$$G_3' = \cos\theta_{31}G_1 + \cos\theta_{32}G_2 + \cos\theta_{33}G_3$$

Orthogonal transformations

- Therefore, **orthogonal transformations** are defined as:

$$G_i' = \sum_{j=1}^{3} a_{ij} G_j; \qquad a_{ij} \equiv \cos\theta_{ij}$$

- Orthogonal transformations can be expressed as a matrix relationship with a **transformation matrix** A

$$\mathbf{G}' = \mathbf{A}\mathbf{G}$$

- With orthogonality conditions imposed on the transformation matrix A

$$\sum_{l=1}^{3} a_{li} a_{lk} = \delta_{ik}$$

Properties of the transformation matrix

- Introducing a matrix **inverse** to the transformation matrix

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{1} \qquad \sum_{l=1}^{3} a_{kl}\bar{a}_{li} = \delta_{ki}$$

- Let us consider a matrix element $a_{ij} = \sum_{k=1}^{3} a_{kj}\delta_{ki}$

$$= \sum_{k=1}^{3}\left(a_{kj}\left(\sum_{l=1}^{3} a_{kl}\bar{a}_{li} \right)\right) = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{kj}a_{kl}\bar{a}_{li} = \sum_{l=1}^{3}\left(\bar{a}_{li}\left(\sum_{k=1}^{3} a_{kj}a_{kl} \right)\right)$$

$$= \sum_{l=1}^{3} \bar{a}_{li}\delta_{jl} = \bar{a}_{ji} = a_{ij}$$

$$\boxed{\mathbf{A}^{-1} = \tilde{\mathbf{A}}}$$

- Orthogonality conditions $\sum_{k=1}^{3} a_{kj}a_{kl} = \delta_{jl}$

Properties of the transformation matrix

$$\tilde{\mathbf{A}} = \mathbf{A}^{-1} \qquad \mathbf{A}\tilde{\mathbf{A}} = \mathbf{A}\mathbf{A}^{-1} \qquad \mathbf{A}\tilde{\mathbf{A}} = \mathbf{1}$$

- Calculating the determinants

$$\left|\mathbf{A}\tilde{\mathbf{A}}\right| = \left|\mathbf{A}\right|\left|\tilde{\mathbf{A}}\right| = \left|\mathbf{A}\right|\left|\mathbf{A}\right| = \left|\mathbf{A}\right|^2 = \left|\mathbf{1}\right| = 1$$

$$\therefore \left|\mathbf{A}\right| = \pm 1$$

- The case of a negative determinant corresponds to a complete inversion of coordinate axes and is **not physical** (a.k.a. **improper**)

Properties of the transformation matrix

- In a general case, there are **9** non-vanishing elements in the transformation matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- In a general case, there are **6** independent equations in the orthogonality conditions

$$\sum_{l=1}^{3} \cos\theta_{li} \cos\theta_{lk} = \delta_{ik}$$

$$\hat{i}'\cdot\hat{j}' = \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = 0$$

$$\hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1$$

- Therefore, there are **3** independent coordinates that describe the orientation of the rigid body

Example: rotation in a plane

- Let's consider a 2D rotation of a position vector *r*

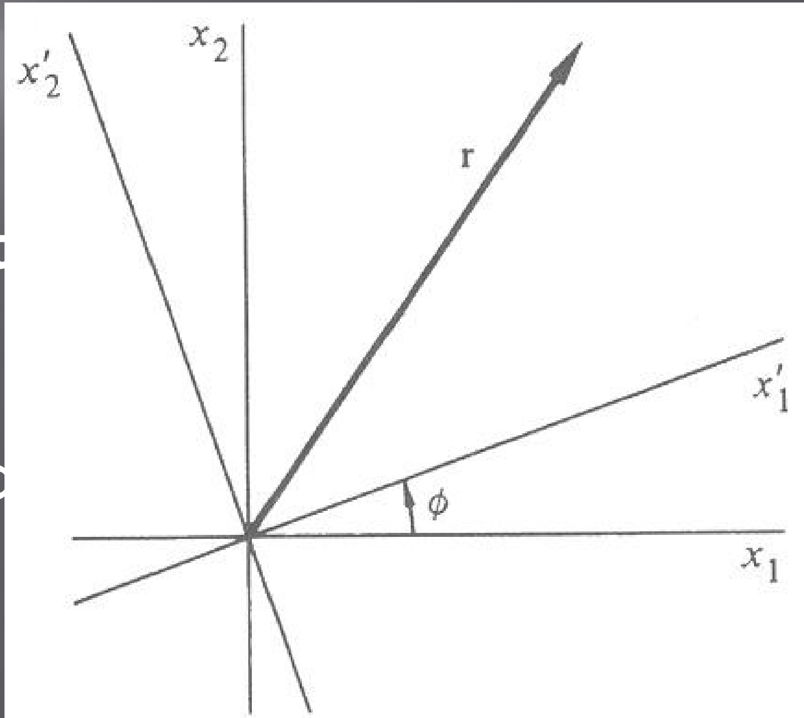
- The z component of the vector is not affected, therefore the transformation matrix should look like

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- With the orthogonality condit

$$\sum_{k=1}^{2} a_{kj} a_{kl} = \delta_{jl} \qquad j,l = 1,2$$

- The total number of independ
  coordinates is
  4 – 3 = 1



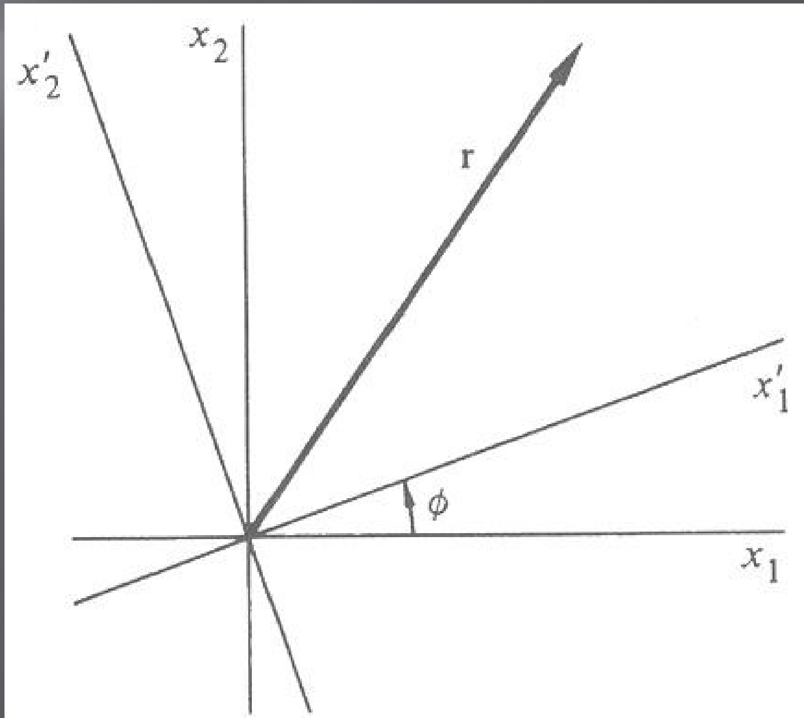
Example: rotation in a plane

- The most natural choice for the independent coordinate would be the **angle of rotation**, so that

$$x_1' = x_1 \cos\phi + x_2 \sin\phi$$
$$x_2' = -x_1 \sin\phi + x_2 \cos\phi$$
$$x_3' = x_3$$

- The transformation matrix

$$\mathbf{A} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



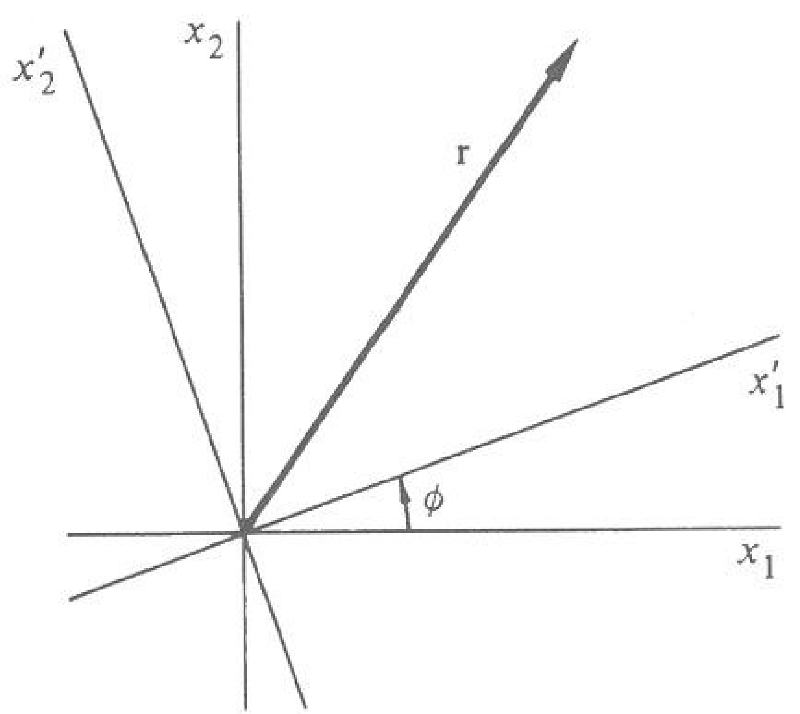
Example: rotation in a plane

- **The three orthogonality conditions**

$$a_{11}a_{11} + a_{21}a_{21} = 1$$
$$a_{12}a_{12} + a_{22}a_{22} = 1$$
$$a_{11}a_{12} + a_{21}a_{22} = 0$$

- **They are rewritten as**

$$\cos^2\phi + \sin^2\phi = 1$$
$$\sin^2\phi + \cos^2\phi = 1$$
$$\cos\phi\sin\phi - \sin\phi\cos\phi = 0$$



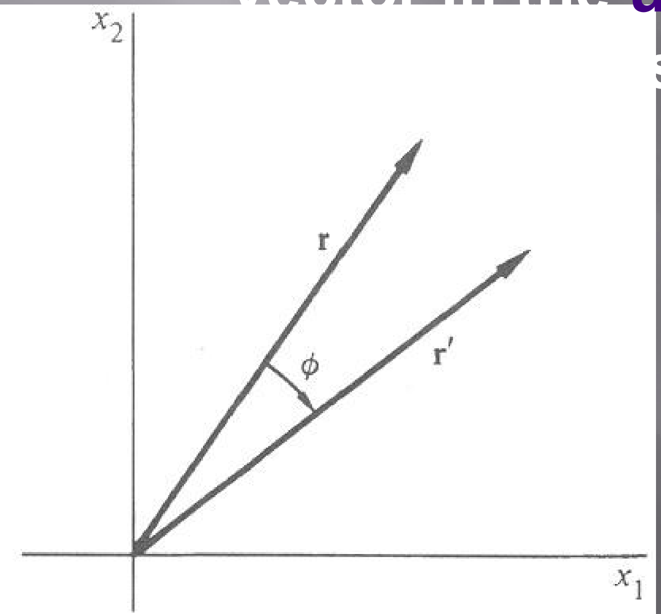
Example: rotation in a plane

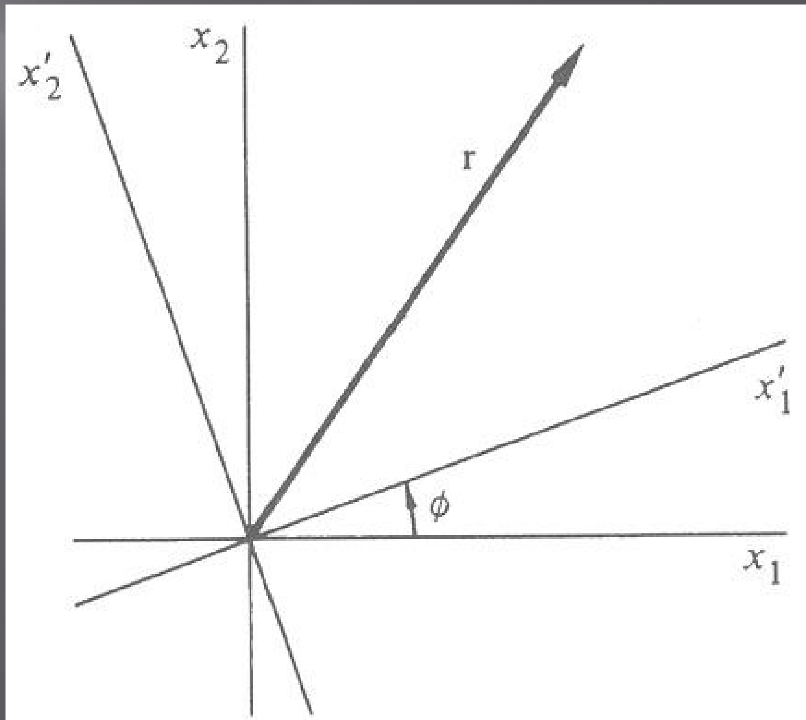
- The 2D transformation matrix

$$\mathbf{A} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- It describes a **CCW** rotation of the coordinate axes

- Alternatively, it can describe a **CW** rotation of the same vector in the **unchanged** … stem





The Euler angles

• In order to describe the motion of rigid bodies in the canonical formulation of mechanics, it is necessary to seek **three** independent parameters that specify the orientation of a rigid body

• The most common and useful set of such parameters are the **Euler angles**

• The Euler angles correspond to an orthogonal transformation via three successive rotations performed in a specific sequence

• The Euler transformation matrix is proper

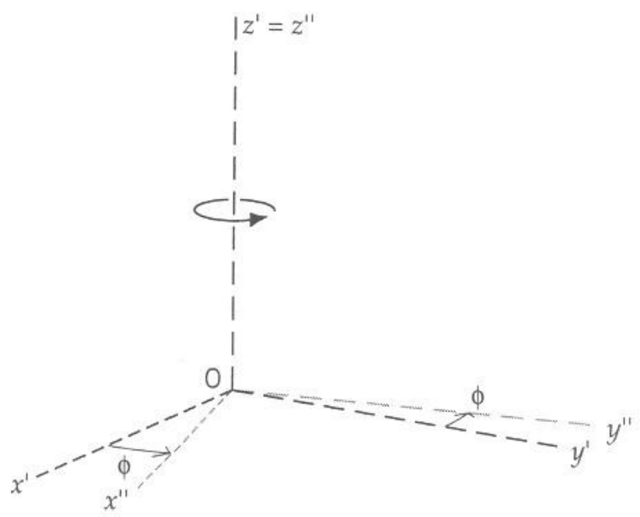
$$|\mathbf{A}| = 1$$

Leonhard Euler
(1707 – 1783)

The Euler angles

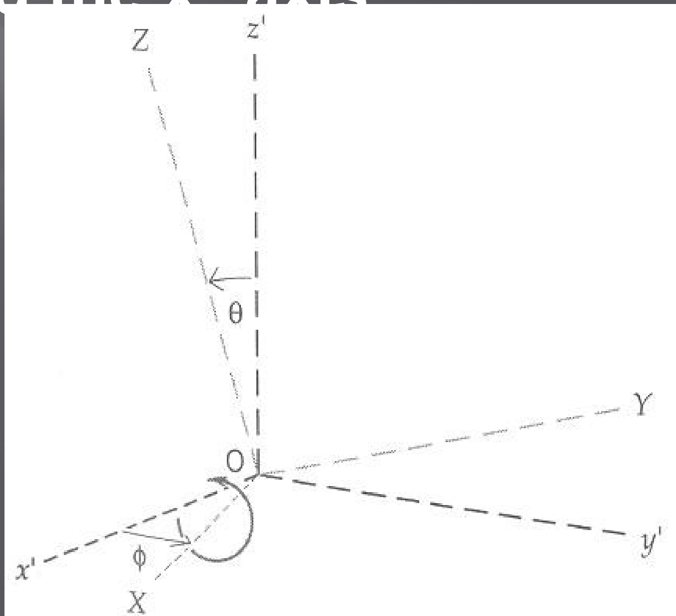
- First, we rotate the system around the z' axis

$$\mathbf{x}'' = \mathbf{D}\mathbf{x}' = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$



- Then we rotate the system around the x'' axis

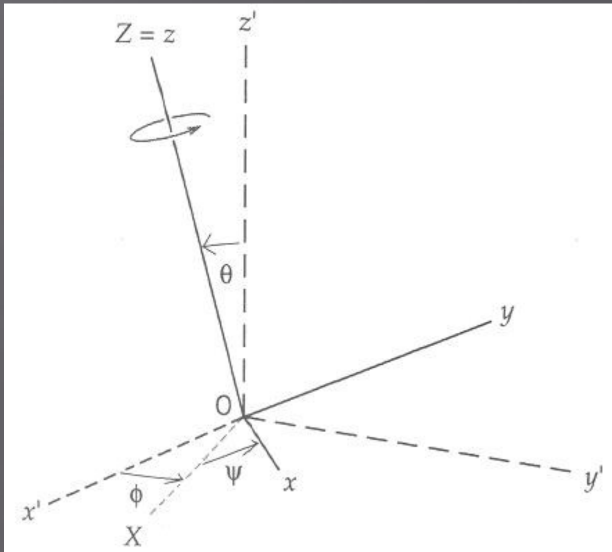
$$\mathbf{X} = \mathbf{C}\mathbf{x}'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$



The Euler angles

• Finally, we rotate the system around the Z axis

$$\mathbf{x} = \mathbf{B}\mathbf{X} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



• The complete transformation can be expressed as a product of the successive matrices

$$\mathbf{x} = \mathbf{B}\mathbf{X} = \mathbf{B}\mathbf{C}\mathbf{x}'' = \mathbf{B}\mathbf{C}\mathbf{D}\mathbf{x}' \equiv \mathbf{A}\mathbf{x}'$$

$$\boxed{\mathbf{x} = \mathbf{A}\mathbf{x}'}$$

The Euler angles

- The explicit form of the resultant transformation matrix A is

$$\mathbf{A} = \mathbf{BCD} =$$

$$= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

- The described sequence is known as the **x-convention**

- Overall, there are **12** different possible conventions in defining the Euler angles

Euler theorem

- **Euler theorem**: the general displacement of a rigid body with one point fixed is a rotation about some axis

- If the fixed point is taken as the origin, then the displacement of the rigid body involves **no translation**; only the **change in orientation**

- If such a rotation could be found, then the axis of rotation would be unaffected by this transformation

- Thus, any vector lying along the axis of rotation must have the same components before and after the orthogonal transformation:

$$\mathbf{R}' = \mathbf{A}\mathbf{R} = \mathbf{R}$$

Euler theorem

$$\mathbf{AR} = \mathbf{R} \qquad \mathbf{AR} = 1\mathbf{R} \qquad (\mathbf{A} - \mathbf{1})\mathbf{R} = 0$$

- This formulation of the Euler theorem is equivalent to an eigenvalue problem $(\mathbf{A} - \lambda\mathbf{1})\mathbf{R} = 0$

- With one of the eigenvalues $\lambda = 1$

- So we have to show that the orthogonal transformation matrix has at least one eignevalue $\lambda = 1$

- The secular equation of an eigenvalue problem is

$$\left|\mathbf{A} - \lambda\mathbf{1}\right| = 0$$

- It can be rewritten for the case of $\lambda = 1$

$$\left|\mathbf{A} - \mathbf{1}\right| = 0$$

Euler theorem

- Recall the orthogonality condition: $|\mathbf{A}| = 1 \qquad \mathbf{A}\tilde{\mathbf{A}} = \mathbf{1}$

$$\mathbf{A}\tilde{\mathbf{A}} - \tilde{\mathbf{A}} = \mathbf{1} - \tilde{\mathbf{A}} \qquad (\mathbf{A}-\mathbf{1})\tilde{\mathbf{A}} = \mathbf{1} - \tilde{\mathbf{A}} \qquad |(\mathbf{A}-\mathbf{1})\tilde{\mathbf{A}}| = |\mathbf{1} - \tilde{\mathbf{A}}|$$

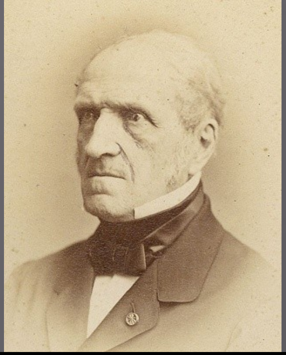
$$|\mathbf{A}-\mathbf{1}||\tilde{\mathbf{A}}| = |\mathbf{1} - \tilde{\mathbf{A}}| \qquad |\mathbf{A}-\mathbf{1}||\mathbf{A}| = |\mathbf{1} - \tilde{\mathbf{A}}| \qquad |\mathbf{A}-\mathbf{1}| = |\mathbf{1} - \tilde{\mathbf{A}}|$$

$$|\mathbf{A}-\mathbf{1}| = |\widetilde{\mathbf{1} - \mathbf{A}}| \qquad |\mathbf{A}-\mathbf{1}| = |\mathbf{1} - \mathbf{A}| \qquad |\mathbf{A}-\mathbf{1}| = |-(\mathbf{A}-\mathbf{1})|$$

$$|\mathbf{A}-\mathbf{1}| = (-1)^n |\mathbf{A}-\mathbf{1}|$$

- *n* is the dimension of the square matrix
- For 3D case: $|\mathbf{A}-\mathbf{1}| = (-1)^3 |\mathbf{A}-\mathbf{1}| \qquad |\mathbf{A}-\mathbf{1}| = -|\mathbf{A}-\mathbf{1}|$
- It can be true only if $|\mathbf{A}-\mathbf{1}| = 0$ *Q.E.D.*

Euler theorem



Michel Chasles (1793–1880)

- **For 2D case (rotation in a plane) *n* = 2:**

$$|\mathbf{A}-\mathbf{1}| = (-1)^n |\mathbf{A}-\mathbf{1}| \qquad |\mathbf{A}-\mathbf{1}| = |\mathbf{A}-\mathbf{1}|$$

- **Euler theorem does not hold for <u>all</u> orthogonal transformation matrices in 2D: there is no vector in the plane of rotation that is left unaltered – only a point**

- **To find the direction of the rotation axis one has to solve the system of equations for three components of vector *R:***

$$(\mathbf{A}-\mathbf{1})\mathbf{R} = 0$$

- **Removing the constraint, we obtain Chasles' theorem: the most general displacement of a rigid body is a translation plus a rotation**

Infinitesimal rotations

- Let us consider orthogonal transformation matrices of the following form

$$\mathbf{A} = \mathbf{1} + \boldsymbol{\alpha}$$

- Here *α* is a square matrix with **infinitesimal** elements

- Such matrices A are called **matrices of infinitesimal rotations**

- Generally, two rotations do not commute

$$\mathbf{A}_1\mathbf{A}_2\mathbf{R} \neq \mathbf{A}_2\mathbf{A}_1\mathbf{R}$$

- Infinitesimal rotations **do** commute

$$(\mathbf{1}+\boldsymbol{\alpha}_1)(\mathbf{1}+\boldsymbol{\alpha}_2) = \mathbf{1}+\boldsymbol{\alpha}_1\mathbf{1}+\mathbf{1}\boldsymbol{\alpha}_2+\cancel{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} = \mathbf{1}+\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2$$

$$(\mathbf{1}+\boldsymbol{\alpha}_2)(\mathbf{1}+\boldsymbol{\alpha}_1) = \mathbf{1}+\boldsymbol{\alpha}_2\mathbf{1}+\mathbf{1}\boldsymbol{\alpha}_1+\cancel{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1} = \mathbf{1}+\boldsymbol{\alpha}_2+\boldsymbol{\alpha}_1$$

Infinitesimal rotations

- The inverse of the infinitesimal rotation: $\mathbf{A}^{-1} = \mathbf{1} - \boldsymbol{\alpha}$
- Proof: $(\mathbf{1}-\boldsymbol{\alpha})\mathbf{A} = (\mathbf{1}-\boldsymbol{\alpha})(\mathbf{1}+\boldsymbol{\alpha}) = \mathbf{1} - \boldsymbol{\alpha}\mathbf{1} + \mathbf{1}\boldsymbol{\alpha} - \boldsymbol{\alpha}\boldsymbol{\alpha} = \mathbf{1}$
- On the other hand: $\mathbf{A}^{-1} = \widetilde{\mathbf{A}}$ $\quad \mathbf{1} - \boldsymbol{\alpha} = \mathbf{1} + \widetilde{\boldsymbol{\alpha}}$ $\quad \therefore \boxed{\widetilde{\boldsymbol{\alpha}} = -\boldsymbol{\alpha}}$
- Matrices *α* are **antisymmetric**
- In 3D we can write:

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$$

- Infinitesimal change of a vector:

$$d\mathbf{r} = \mathbf{r}' - \mathbf{r} = (\mathbf{1}+\boldsymbol{\alpha})\mathbf{r} - \mathbf{r} = \boldsymbol{\alpha}\mathbf{r} \qquad (dr)_i = \sum_{j=1}^{3} \alpha_{ij} r_j$$

Infinitesimal rotations

$$(dr)_i = \sum_{j=1}^{3} \alpha_{ij} r_j$$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$$

$$(dr)_1 = r_2 d\Omega_3 - r_3 d\Omega_2$$

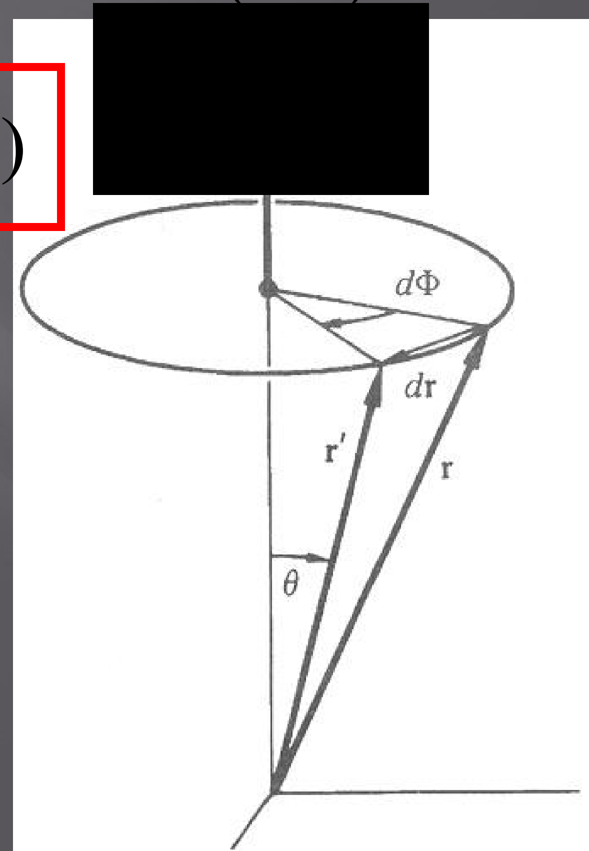
$$(dr)_2 = r_3 d\Omega_1 - r_1 d\Omega_3$$

$$(dr)_3 = r_1 d\Omega_2 - r_2 d\Omega_1$$

$$(dr)_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} r_j d\Omega_k$$

$$\overrightarrow{(d\Omega)} = \vec{n}\, d\Phi$$

$$d\vec{r} = \vec{r} \times \overrightarrow{(d\Omega)}$$



- $\overrightarrow{(d\Omega)}$ is a differential vector, <u>not</u> a differential <u>of a</u> vector

$$dr = (r \sin\theta)\, d\Phi \qquad d\vec{r} = (\vec{r} \times \vec{n})\, d\Phi$$

- $\overrightarrow{(d\Omega)}$ is **normal** to the rotation plane

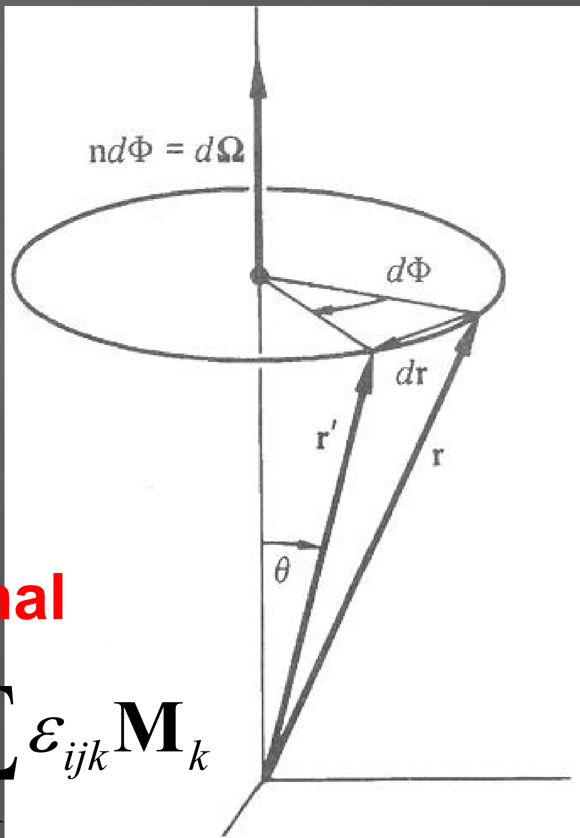
Infinitesimal rotations

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{bmatrix} d\Phi$$

$$(\overrightarrow{d\Omega}) = \vec{n}\, d\Phi$$

$$\boldsymbol{\alpha} = d\Phi \sum_{i=1}^{3} n_i \mathbf{M}_i \qquad \mathbf{M}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \mathbf{M}_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



- These matrices are called **infinitesimal rotation generators**

$$\mathbf{M}_i \mathbf{M}_j - \mathbf{M}_j \mathbf{M}_i = \sum_{i=1}^{3} \varepsilon_{ijk} \mathbf{M}_k$$

Example: infinitesimal Euler angles

$$\mathbf{A} =$$

$$= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

- For infinitesimal Euler angles it can be rewritten as

$$\mathbf{A} = \begin{bmatrix} 1 & d\phi + d\psi & 0 \\ -(d\phi + d\psi) & 1 & d\theta \\ 0 & -d\theta & 1 \end{bmatrix} = \mathbf{1} + \boldsymbol{\alpha}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & d\phi + d\psi & 0 \\ -(d\phi + d\psi) & 0 & d\theta \\ 0 & -d\theta & 0 \end{bmatrix}$$

$$(d\vec{\Omega}) = \hat{i}\,d\theta + \hat{k}(d\phi + d\psi)$$

Rate of change of a vector

$$\mathbf{G}' = \mathbf{A}\mathbf{G} \qquad G_i' = \sum_{j=1}^{3} a_{ij} G_j \qquad a_{ij} = \delta_{ij} + da_{ij} = \delta_{ij} + \alpha_{ij}$$

$$dG_i' = \sum_{j=1}^{3}\left(a_{ij} dG_j + G_j da_{ij}\right) = \sum_{j=1}^{3}\left((\delta_{ij} + da_{ij}) dG_j + G_j da_{ij}\right)$$

$$= \sum_{j=1}^{3}\left(\delta_{ij} dG_j + G_j \alpha_{ij}\right) = dG_i + \sum_{j=1}^{3}\left(G_j \sum_{k=1}^{3} \varepsilon_{ijk} d\Omega_k\right) = dG_i + \sum_{j,k=1}^{3} \varepsilon_{ijk} G_j d\Omega_k$$

$$\alpha_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk} d\Omega_k$$

$$dG_i' = dG_i + \left(\mathbf{G} \times d\boldsymbol{\Omega}\right)_i$$

- Dividing by *dt*

$$\frac{d\mathbf{G}'}{dt} = \frac{d\mathbf{G}}{dt} + \mathbf{G} \times \boldsymbol{\omega} \qquad \boldsymbol{\omega} dt \equiv d\boldsymbol{\Omega} \qquad \dot{\mathbf{G}}' = \dot{\mathbf{G}} + \mathbf{G} \times \boldsymbol{\omega}$$

Example: the Coriolis effect

$$\dot{\mathbf{G}} = \dot{\mathbf{G}}' + \boldsymbol{\omega} \times \mathbf{G}$$

$$\vec{\omega} = const$$

• Velocity vectors in the rotating and in the "stationary" systems are related as

$$\dot{\vec{r}}_s = \dot{\vec{r}}_r + \vec{\omega} \times \vec{r} \qquad \vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r}$$

• For the rate of change of velocity

$$(\dot{\vec{v}}_s)_s = (\dot{\vec{v}}_s)_r + \vec{\omega} \times \vec{v}_s = \left(\frac{d(\vec{v}_r + \vec{\omega} \times \vec{r})}{dt} \right)_r + \vec{\omega} \times (\vec{v}_r + \vec{\omega} \times \vec{r})$$

$$= (\dot{\vec{v}}_r)_r + 2\vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a}_r = \vec{a}_s - 2\vec{\omega} \times \vec{v}_r - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

• Rotating system: acceleration acquires Coriolis and centrifugal components

Gaspard-Gustave Coriolis (1792 - 1843)

Example: the Coriolis effect

$$\vec{a}_c = \vec{v}_r \times 2\vec{\omega}$$

- On the other hand

$$\vec{a}_L = \vec{v} \times \frac{q\vec{B}}{m}$$

- This is the **Lorentz acceleration**

$$\vec{a}_L = \vec{v} \times \vec{\omega}_L \qquad \vec{\omega}_L \equiv \frac{q\vec{B}}{m}$$

- What is the relationship between those two?

Kinetic energy of a system of particles

- **Kinetic energy of a system of particles** $T=\frac{1}{2}\sum_i m_i(\dot{r}_i)^2$

- **Introducing a center of mass:**

$$\vec{R}=\frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}=\frac{\sum_i m_i\vec{r}_i}{M}$$

$$\sum_i m_i\vec{r}_i = M\vec{R}$$

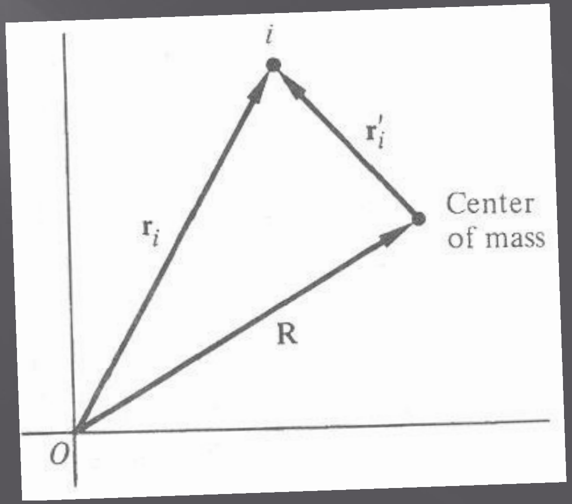
- **We can rewrite the coordinates in the center-of-mass coordinate system:**

$$\vec{r}_i=\vec{r}_i'+\vec{R} \qquad \dot{\vec{r}}_i=\dot{\vec{r}}_i'+\dot{\vec{R}}$$

- **Kinetic energy can be rewritten:**

$$T=\frac{1}{2}\sum_i m_i(\dot{r}_i)^2=\frac{1}{2}\sum_i m_i(\dot{\vec{r}}_i'+\dot{\vec{R}})\cdot(\dot{\vec{r}}_i'+\dot{\vec{R}})$$

$$=\frac{1}{2}\sum_i m_i(\dot{r}_i')^2+\sum_i m_i(\dot{\vec{r}}_i'\cdot\dot{\vec{R}})+\frac{1}{2}\sum_i m_i(\dot{R})^2$$



Kinetic energy of a system of particles

$$T = \frac{1}{2}\sum_i m_i(\dot{r}_i')^2 + \sum_i m_i(\dot{\vec{r}}_i'\cdot\dot{\vec{R}}) + \frac{1}{2}\sum_i m_i(\dot{R})^2$$

$$= \frac{1}{2}\sum_i m_i(\dot{r}_i')^2 + \dot{\vec{R}}\cdot\sum_i m_i\dot{\vec{r}}_i' + \frac{1}{2}(\dot{R})^2\sum_i m_i$$

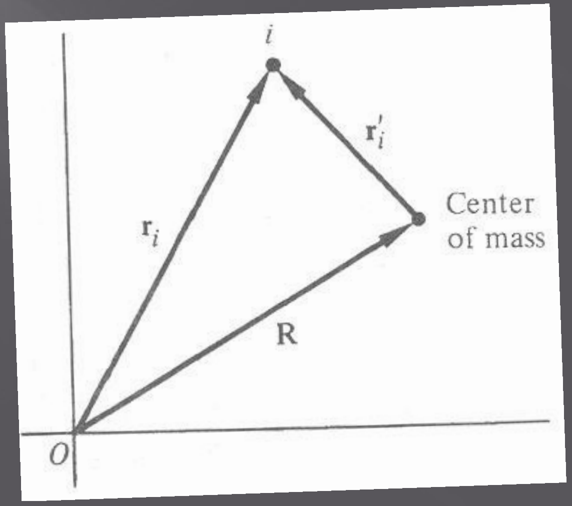
$$= \frac{1}{2}\sum_i m_i(\dot{r}_i')^2 + \dot{\vec{R}}\cdot\frac{d}{dt}\sum_i m_i\vec{r}_i' + \frac{1}{2}(\dot{R})^2 M$$

- On the other hand

$$\sum_i m_i\vec{r}_i = M\vec{R} \qquad \sum_i m_i\vec{r}_i' = M\vec{R}'$$

- In the center-of-mass coordinate system, the center of mass is at the origin, therefore

$$T = \frac{1}{2}\sum_i m_i(\dot{r}_i')^2 + \frac{1}{2}(\dot{R})^2 M$$



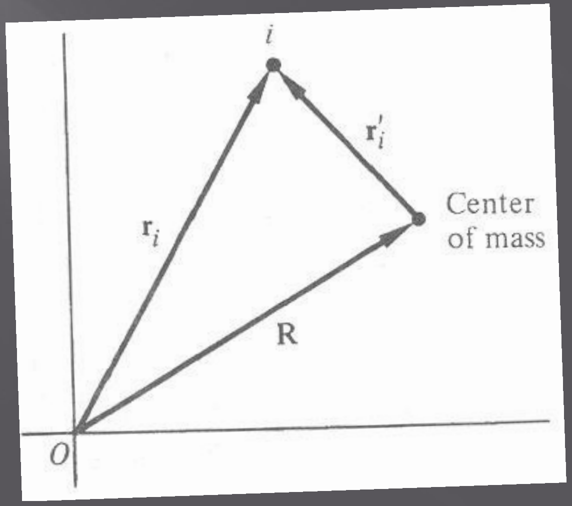
Kinetic energy of a system of particles

- Kinetic energy of the system of particles consists of a kinetic energy about the center of mass plus a kinetic energy obtained if all the mass were concentrated at the center of mass

- This statement can be applied to the case of a **rigid body**: Kinetic energy of a rigid body consists of a kinetic energy about the center of mass plus a kinetic energy obtained if all the mass were concentrated at the center of mass

- Recall Chasles' theorem!

$$T = \frac{1}{2}\sum_i m_i (\dot{r}_i')^2 + \frac{1}{2}(\dot{R})^2 M$$



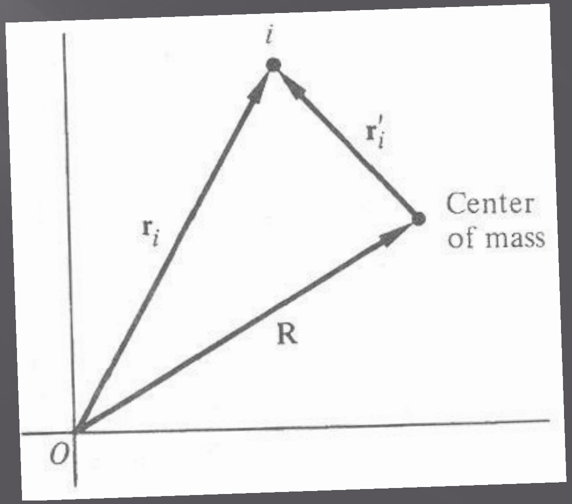
Kinetic energy of a system of particles

- Chasles: we can represent motion of a rigid body as a combination of a **rotation** and **translation**

- If the potential and/or the generalized external forces are known, the translational motion of center of mass can be dealt with separately, as a motion of a **point object**

- Let us consider the rotational part or motion

$$T_R = \frac{1}{2}\sum_i m_i (\dot{r}_i')^2$$



$$T = \frac{1}{2}\sum_i m_i (\dot{r}_i')^2 + \frac{1}{2}(\dot{R})^2 M$$

Rotational kinetic energy

$$T_R = \frac{1}{2}\sum_i m_i(\dot{r}_i')^2 = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i' \cdot \dot{\vec{r}}_i' = \frac{1}{2}\sum_i m_i (\vec{\omega}\times\vec{r}_i')\cdot(\vec{\omega}\times\vec{r}_i')$$

- **Rate of change of a vector** $(\dot{\vec{r}}_i')_s = \cancel{(\dot{\vec{r}}_i')_r} + \vec{\omega}\times\vec{r}_i'$

- **For a rigid body, in the rotating frame of reference, all the distances between the points of the rigid body are fixed:** $(\dot{\vec{r}}_i')_r = 0 \qquad \therefore (\dot{\vec{r}}_i')_s = \vec{\omega}\times\vec{r}_i'$

- **Rotational kinetic energy:**

$$T_R = \frac{1}{2}\sum_i m_i (\vec{\omega}\times\vec{r}_i')\cdot(\vec{\omega}\times\vec{r}_i') = \frac{1}{2}\sum_i m_i \sum_{j=1}^{3} (\vec{\omega}\times\vec{r}_i')_j \cdot (\vec{\omega}\times\vec{r}_i')_j$$

$$= \frac{1}{2}\sum_i m_i \sum_{j=1}^{3}\left(\left(\sum_{k,l=1}^{3} \varepsilon_{jkl}\omega_k r_i{}'_l\right)\left(\sum_{m,n=1}^{3} \varepsilon_{jmn}\omega_m r_i{}'_n\right)\right)$$

Rotational kinetic energy

$$T_R = \frac{1}{2}\sum_i m_i \sum_{j=1}^{3}\left(\left(\sum_{k,l=1}^{3}\varepsilon_{jkl}\omega_k r_i{}'_l\right)\left(\sum_{m,n=1}^{3}\varepsilon_{jmn}\omega_m r_i{}'_n\right)\right)$$

$$\sum_{j=1}^{3}\varepsilon_{jkl}\varepsilon_{jmn} = \\ = \delta_{km}\delta_{ln} - \delta_{lm}\delta_{kn}$$

$$= \frac{1}{2}\sum_i \sum_{j,k,l,m,n=1}^{3} m_i \varepsilon_{jkl}\varepsilon_{jmn}\omega_k\omega_m r_i{}'_l\, r_i{}'_n$$

$$= \frac{1}{2}\sum_i \sum_{k,l,m,n=1}^{3} m_i(\delta_{km}\delta_{ln} - \delta_{lm}\delta_{kn})\omega_k\omega_m r_i{}'_l\, r_i{}'_n$$

$$= \frac{1}{2}\sum_i m_i\left(\sum_{k,l=1}^{3}(\omega_k)^2(r_i{}'_l)^2 - \sum_{k,l=1}^{3}\omega_k r_i{}'_k\, r_i{}'_l\,\omega_l\right)$$

$$= \frac{1}{2}\sum_{k,l=1}^{3}\omega_k\omega_l\sum_i m_i[(r_i{}')^2\delta_{kl} - r_i{}'_k\, r_i{}'_l] = \frac{1}{2}\sum_{k,l=1}^{3}\omega_k I_{kl}\omega_l = \frac{\tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}}{2}$$

$$I_{kl} \equiv \sum_i m_i[(r_i{}')^2\delta_{kl} - r_i{}'_k\, r_i{}'_l]$$

Inertia tensor and moment of inertia

$$T_R = \frac{\tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}}{2}$$

$$I_{kl} \equiv \sum_i m_i[(r_i')^2\delta_{kl} - r'_{i\,k}\, r'_{i\,l}\,]$$

- (3x3) matrix **I** is called the inertia tensor

- Inertia tensor is a symmetric matrix (only 6 independent elements):

$$I_{kl} = I_{lk}$$

- For a rigid body with a continuous distribution of density, the definition of the inertia tensor is as follows:

$$I_{kl} \equiv \int_V \rho[(r)^2\delta_{kl} - r_k r_l]dV$$

- Introducing a notation

$$\boldsymbol{\omega} = \omega\mathbf{n}$$

$$T_R = \frac{\tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}}{2} = \frac{\omega\tilde{\mathbf{n}}\mathbf{I}\mathbf{n}\omega}{2} = \frac{I\omega^2}{2}$$

$$I \equiv \tilde{\mathbf{n}}\mathbf{I}\mathbf{n}$$

- Scalar *I* is called the moment of inertia

Inertia tensor and moment of inertia

$$T_R = \frac{I\omega^2}{2}$$

- **On the other hand:**

$$T_R = \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r}_i')\cdot(\vec{\omega}\times\vec{r}_i') = \frac{\omega^2}{2}\sum_i m_i(\vec{n}\times\vec{r}_i')\cdot(\vec{n}\times\vec{r}_i')$$

- **Therefore**

$$I = \sum_i m_i(\vec{n}\times\vec{r}_i')\cdot(\vec{n}\times\vec{r}_i')$$

- **The moment of inertia depends upon the position and direction of the axis of rotation**

Parallel axis theorem

- For a constrained rigid body, the rotation may occur not around the center of mass, but around some other point 0, fixed at a given moment of time

- Then, the moment of inertia about the axis of rotation is:

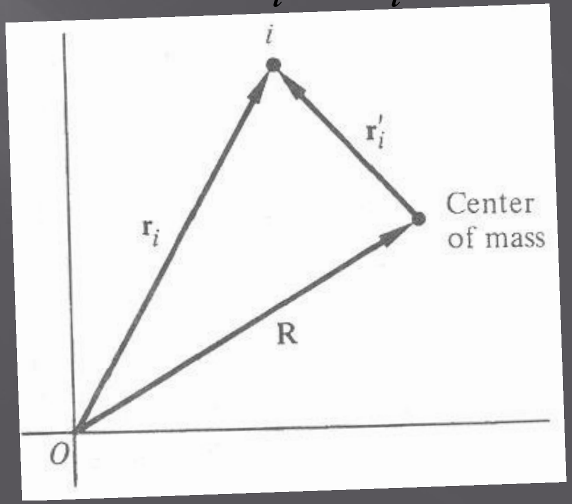
$$I_0 = \sum_i m_i (\vec{n} \times \vec{r}_i) \cdot (\vec{n} \times \vec{r}_i) = \sum_i m_i (\vec{n} \times (\vec{r}_i' + \vec{R})) \cdot (\vec{n} \times (\vec{r}_i' + \vec{R}))$$

$$\vec{r}_i = \vec{r}_i' + \vec{R}$$

$$= \sum_i m_i (\vec{n} \times \vec{r}_i')^2 + 2 \sum_i m_i (\vec{n} \times \vec{r}_i') \cdot (\vec{n} \times \vec{R})$$

$$+ \sum_i m_i (\vec{n} \times \vec{R})^2 \quad = I_{CM}$$

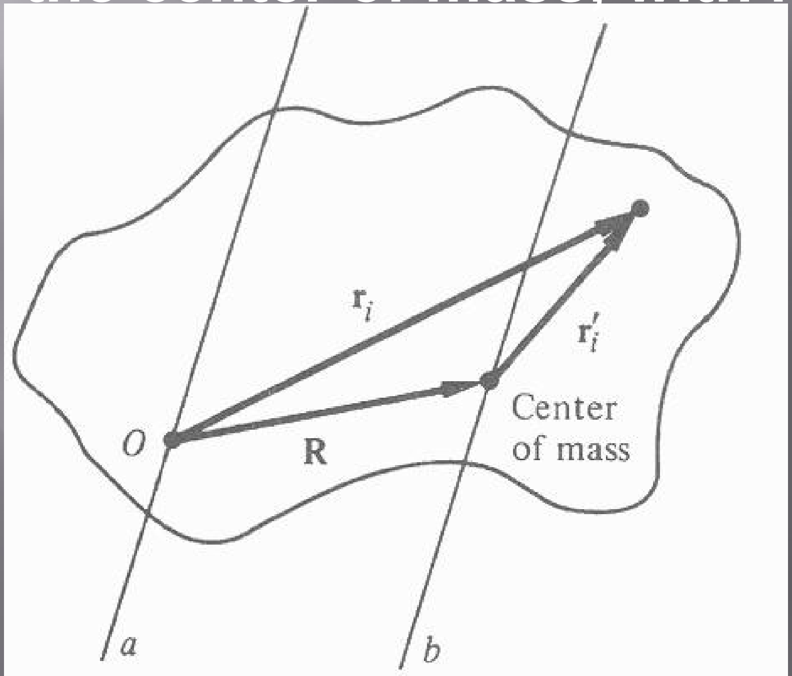
$$+ 2(\vec{n} \times \sum_i m_i \vec{r}_i') \cdot (\vec{n} \times \vec{R}) + (\vec{n} \times \vec{R})^2 M$$

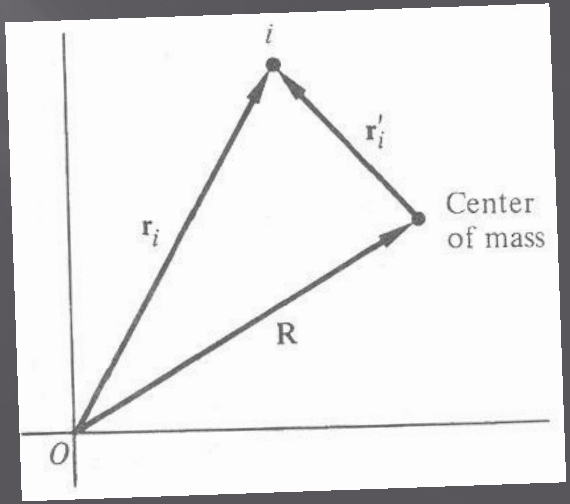


Parallel axis theorem

$$I_0 = I_{CM} + M(\vec{n}\times\vec{R})^2$$

• **Parallel axis theorem**: the moment of inertia about a given axis is equal to the moment of inertia about a parallel axis through the center of mass plus the moment of inertia of the body, as if concentrated at the center of mass, with respect to the original axis





Parallel axis theorem

- Does the change of axes affect the $\boldsymbol{\omega}$ vector?

- Let us consider two systems of coordinates defined with respect to two **different** points of the rigid body: $x'_1y'_1z'_1$ and $x'_2y'_2z'_2$

$$\vec{R}_2 = \vec{R}_1 + \vec{R}$$

- Then $(\dot{\vec{R}}_2)_s = (\dot{\vec{R}}_1)_s + (\dot{\vec{R}})_s = (\dot{\vec{R}}_1)_s + \cancel{(\dot{\vec{R}})_r} + \vec{\omega}_1 \times \vec{R}$

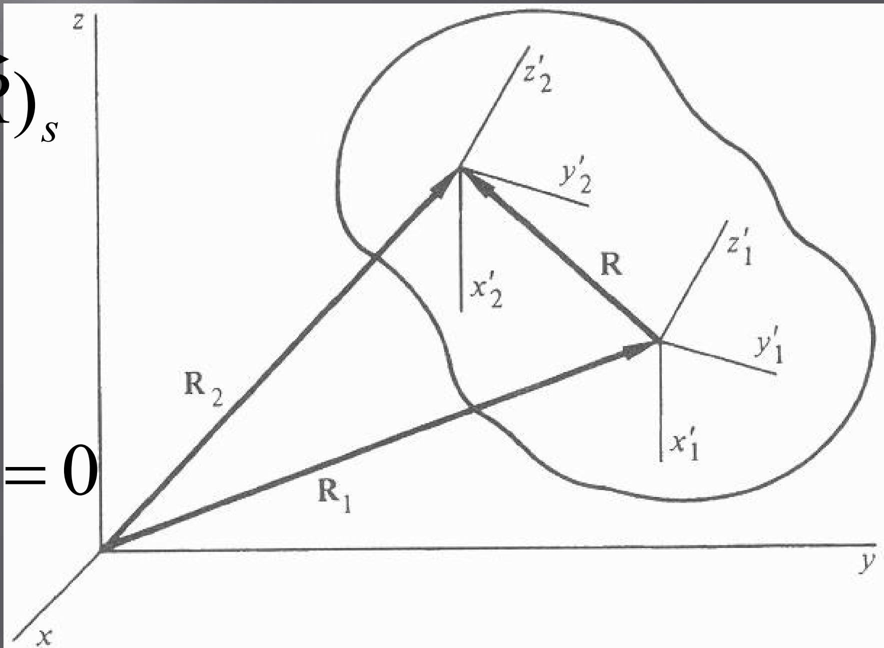
- Similarly $(\dot{\vec{R}}_1)_s = (\dot{\vec{R}}_2)_s - (\dot{\vec{R}})_s$

$$= (\dot{\vec{R}}_2)_s - \cancel{(\dot{\vec{R}})_r} - \vec{\omega}_2 \times \vec{R}$$

$$(\dot{\vec{R}}_2)_s = (\dot{\vec{R}}_1)_s + \vec{\omega}_1 \times \vec{R}$$

$$(\vec{\omega}_1 - \vec{\omega}_2) \times \vec{R} = 0$$

$$(\dot{\vec{R}}_1)_s = (\dot{\vec{R}}_2)_s - \vec{\omega}_2 \times \vec{R}$$



Parallel axis theorem

$$(\vec{\omega}_1 - \vec{\omega}_2) \times \vec{R} = 0$$

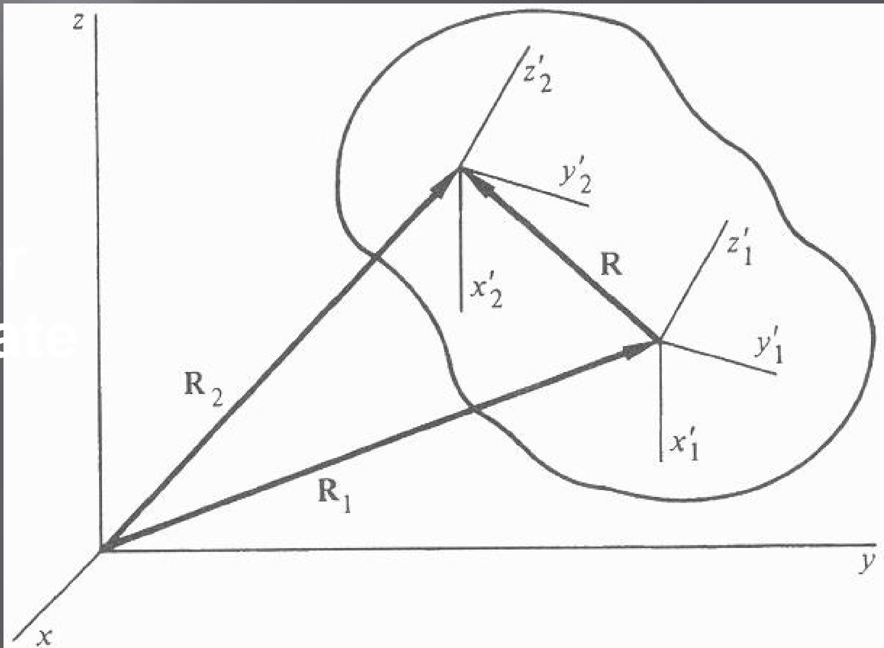
- **Any** difference in ω vectors at two **arbitrary** points must be parallel to the line joining two points

- It is not possible for **all the points** of the rigid body

- Then, the only possible case:

$$\vec{\omega}_1 = \vec{\omega}_2$$

- The angular velocity vector is **the same** for all coordinate systems fixed in the body



Example: inertia tensor of a homogeneous cube

- Let us consider a homogeneous cube of mass *M* and side *a*

- Let us choose the origin at one of cube's corners

- Then $$I_{kl} = \int_V \rho[(r)^2 \delta_{kl} - r_k r_l]dV$$

$$I_{11} = \rho \int_0^a \int_0^a \int_0^a [(r)^2 - r_1 r_1] dr_1 dr_2 dr_3 = \rho \int_0^a \int_0^a \int_0^a [(r_2)^2 + (r_3)^2] dr_1 dr_2 dr_3$$

$$= \rho a \int_0^a \int_0^a [(r_2)^2 + (r_3)^2] dr_2 dr_3 = \frac{2\rho a^5}{3} = \frac{2Ma^2}{3} = I_{22} = I_{33}$$

Example: inertia tensor of a homogeneous cube

$$I_{kl} = \int_V \rho[(r)^2 \delta_{kl} - r_k r_l] dV$$

$$I_{12} = \rho \int_0^a \int_0^a \int_0^a [-r_1 r_2] dr_1 dr_2 dr_3 = -\rho a \int_0^a \int_0^a [r_1 r_2] dr_1 dr_2 = -\frac{\rho a^5}{4} = -\frac{Ma^2}{4}$$

$$I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32}$$

$$\mathbf{I} = Ma^2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix}$$

Angular momentum of a rigid body

- **Angular momentum** of a system of particles is:

$$\vec{L} = \sum_i m_i (\vec{r}_i \times \dot{\vec{r}}_i)$$

- Rate of change of a vector $(\dot{\vec{r}}_i)_s = (\dot{\vec{r}}_i)_r + \vec{\omega} \times \vec{r}_i$

- For a **rigid body**, in the rotating frame of reference, all the distances between the points of the rigid body are **fixed**: $(\dot{\vec{r}}_i)_r = 0 \qquad \therefore (\dot{\vec{r}}_i)_s = \vec{\omega} \times \vec{r}_i$

- Angular momentum of rigid body: $\vec{L} = \sum_i m_i (\vec{r}_i \times (\vec{\omega} \times \vec{r}_i))$

$$L_j = \sum_i m_i \left(\sum_{k,l=1}^{3} \varepsilon_{jkl} r_{ik} \left(\sum_{m,n=1}^{3} \varepsilon_{lmn} \omega_m r_{in} \right) \right)$$

$$= \sum_i \sum_{k,l,m,n=1}^{3} \varepsilon_{jkl} \varepsilon_{lmn} r_{ik} r_{in} \omega_m m_i$$

Angular momentum of a rigid body

$$L_j = \sum_i \sum_{k,l,m,n=1}^{3} \varepsilon_{jkl}\varepsilon_{lmn} r_{ik} r_{in} \omega_m m_i$$

$$= \sum_i \sum_{k,m,n=1}^{3} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) r_{ik} r_{in} \omega_m m_i$$

$$= \sum_{k=1}^{3} \omega_k \sum_i m_i [(r_i)^2 \delta_{jk} - r_{ij} r_{ik}] = \sum_{k=1}^{3} I_{jk}\omega_k$$

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

- **Rotational kinetic energy:**

$$T_R = \frac{\tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}}{2} = \frac{\tilde{\boldsymbol{\omega}}\mathbf{L}}{2} = \frac{\tilde{\mathbf{L}}\boldsymbol{\omega}}{2}$$

Free rigid body

- For a free rigid body, the Lagrangian is:

$$L = T = \frac{1}{2}\sum_i m_i (\dot{r}_i')^2 + \frac{1}{2}(\dot{R})^2 M = \frac{1}{2}\sum_{k,l=1}^{3}\omega_k I_{kl}\omega_l + \frac{1}{2}(\dot{R})^2 M$$

- Recall $\omega_i dt = d\Omega_i$
- Then $L = \frac{1}{2}\sum_{k,l=1}^{3}\dot{\Omega}_k I_{kl}\dot{\Omega}_l + T_{CM}$
- We separate the Lagrangian into two independent parts and consider the rotational part separately
- Then, the equations of motion for rotation

$$\frac{d}{dt}\left(\frac{\partial L_R}{\partial \dot{\Omega}_i}\right) = \frac{\partial L_R}{\partial \Omega_i} \qquad \frac{d}{dt}\left(\sum_{k=1}^{3} I_{ik}\dot{\Omega}_k\right) = 0 \qquad \frac{d}{dt}\left(\sum_{k=1}^{3} I_{ik}\omega_k\right) = 0$$

Free rigid body

$$\frac{d}{dt}\left(\sum_{k=1}^{3} I_{ik}\omega_k\right) = 0 \qquad \frac{dL_i}{dt} = 0 \qquad \boxed{\frac{d\vec{L}}{dt} = 0}$$

- Angular momentum of a free rigid body is constant

- In the system of coordinates fixed with the rotating rigid body, the tensor of inertia is a constant – it is often convenient to rewrite the equations of motion in the rotating frame of reference:

$$\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_r + \vec{\omega}\times\vec{L} = 0 \qquad \frac{d}{dt}\left(\sum_{k=1}^{3} I_{ik}\omega_k\right) + \sum_{k,l=1}^{3}\varepsilon_{ikl}\omega_k L_l = 0$$

$$\boxed{\sum_{k=1}^{3} I_{ik}\dot{\omega}_k + \sum_{k,l,m=1}^{3}\varepsilon_{ikl} I_{ml}\omega_k\omega_l = 0}$$

Principal axes of inertia

• Inertia tensor is a symmetric matrix

• In a general case, such matrices can be **diagonalized** – we are looking for a system of coordinates fixed to a rigid body, in which the inertia tensor has a form:

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

• To diagonalize the inertia tensor, we have to find the solutions of a **secular equation**

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0$$

Principal axes of inertia

• Coordinate axes, in which the inertia tensor is diagonal, are called the **principal axes** of a rigid body; the eigenvalues of the secular equations are the components of the **principal moment of inertia**

• After diagonalization of the inertia tensor, the equations of motion for rotation of a free rigid body look like

$$I_i \dot{\omega}_i + \sum_{j,k=1}^{3} \varepsilon_{ijk} \omega_j \omega_k I_k = 0$$

• After diagonalization of the inertia tensor, the rotational kinetic energy a rigid body looks like

$$T_R = \frac{1}{2} \sum_{i=1}^{3} I_i \omega_i^{\,2}$$

Principal axes of inertia

- To find the directions of the principal axes we have to find the directions for the eigenvectors *ω*

- When the rotation occurs around one of the principal axes I_n, there is only one non-zero component ω_n

- In this case, the angular momentum has only one component

$$L_k = I_k \omega_k \delta_{kn}$$

- In this case, the rotational kinetic energy has only one term

$$T_R = \frac{1}{2}\sum_{i=1}^{3} \delta_{in} I_i \omega_i^2 = \frac{I_n \omega_n^2}{2}$$

Stability of a free rotational motion

- Let us choose the body axes along the principal axes of a **free** rotating rigid body

- Let us assume that the rotation axis is **slightly off** the direction of one of the principal axes (α - small parameter):

$$\vec{\omega} = \omega_1 \hat{i}_1 + \alpha \nu_2 \hat{i}_2 + \alpha \nu_3 \hat{i}_3$$

- Then, the equations of motion $I_i \dot{\omega}_i + \sum_{j,k=1}^{3} \varepsilon_{ijk} \omega_j \omega_k I_k = 0$

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = 0$$
$$I_2 \dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3) = 0$$
$$I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = 0$$

$$I_1 \dot{\omega}_1 + \alpha \nu_2 \alpha \nu_3 (I_3 - I_2) = 0$$
$$I_2 \alpha \dot{\nu}_2 + \omega_1 \alpha \nu_3 (I_1 - I_3) = 0$$
$$I_3 \alpha \dot{\nu}_3 + \omega_1 \alpha \nu_2 (I_2 - I_1) = 0$$

Stability of a free rotational motion

$$I_1\dot{\omega}_1 + \alpha\nu_2\alpha\nu_3(I_3 - I_2) = 0$$

$$I_2\alpha\dot{\nu}_2 + \omega_1\alpha\nu_3(I_1 - I_3) = 0$$

$$I_3\alpha\dot{\nu}_3 + \omega_1\alpha\nu_2(I_2 - I_1) = 0$$

$$\dot{\omega}_1 = 0$$

$$\dot{\nu}_2 + \omega_1\nu_3\frac{I_1 - I_3}{I_2} = 0$$

$$\dot{\nu}_3 + \omega_1\nu_2\frac{I_2 - I_1}{I_3} = 0$$

$$\omega_1 = const$$

$$\ddot{\nu}_{2(3)} + \nu_{2(3)}\frac{\omega_1^2}{I_2 I_3}(I_1 - I_2)(I_1 - I_3) = 0$$

$$\ddot{\nu}_{2(3)} + \nu_{2(3)}K = 0 \qquad K \equiv \frac{\omega_1^2}{I_2 I_3}(I_1 - I_2)(I_1 - I_3)$$

Stability of a free rotational motion

$$\ddot{\nu}_{2(3)} + \nu_{2(3)} K = 0 \qquad K \equiv \frac{\dot{\omega}_1^{\,2}}{I_2 I_3}(I_1 - I_2)(I_1 - I_3)$$

• The behavior of solutions of this equation depends on the relative values of the principal moments of inertia

$$I_1 < I_2; I_1 < I_3 \rightarrow$$
$$I_1 > I_2; I_1 > I_3 \rightarrow$$
$$K > 0 \qquad K \equiv \beta^2$$

$$\ddot{\nu}_{2(3)} + \beta^2 \nu_{2(3)} = 0$$

$$\nu_{2(3)} = A_{2(3)} \cos(\beta t + \varphi_{2(3)})$$

• Always stable

$$I_3 < I_1 < I_2 \rightarrow$$
$$I_2 < I_1 < I_3 \rightarrow$$
$$K < 0 \qquad K \equiv -\gamma^2$$

$$\ddot{\nu}_{2(3)} - \gamma^2 \nu_{2(3)} = 0$$

$$\nu_{2(3)} = A_{2(3)} e^{-\gamma t}$$

$$\nu_{2(3)} = A_{2(3)} e^{\gamma t}$$

• Exponentially unstable

Classification of tops

- **Depending on the relative values of the principle values of inertia, rigid body can be classified as follows:**

- **Asymmetrical top**: $I_1 \neq I_2 \neq I_3$

- **Symmetrical top**: $I_1 = I_2 \neq I_3$

- **Spherical top**: $I_1 = I_2 = I_3$

- **Rotator**: $I_1 = I_2 \neq 0; I_3 = 0$

Example: principal axes of a uniform cube

- Previously, we have found the inertia tensor for a uniform cube with the origin at one of the corners, and the coordinate axes along the edges:

$$\mathbf{I} = Ma^2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix}$$

$$\begin{vmatrix} \frac{2Ma^2}{3} - I & -\frac{Ma^2}{4} & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & \frac{2Ma^2}{3} - I & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & -\frac{Ma^2}{4} & \frac{2Ma^2}{3} - I \end{vmatrix} = 0$$

- The secular equation:

$$\left[\frac{11Ma^2}{12} - I\right]\left[\left(\frac{2Ma^2}{3} - I\right)^2 - \frac{M^2a^4}{8} - \frac{Ma^2}{4}\left(\frac{2Ma^2}{3} - I\right)\right] = 0$$

Example: principal axes of a uniform cube

$$\left[\frac{11Ma^2}{12}-I\right]\left[\left(\frac{2Ma^2}{3}-I\right)^2-\frac{M^2a^4}{8}-\frac{Ma^2}{4}\left(\frac{2Ma^2}{3}-I\right)\right]=0$$

$$I_1=\frac{11Ma^2}{12} \qquad I_2=\frac{11Ma^2}{12};I_3=\frac{Ma^2}{6}$$

- To find the directions of the principal axes we have to find the directions for the eigenvectors
- Let us consider $I_3=\frac{Ma^2}{6}$

$$\mathbf{I}\boldsymbol{\omega}_3=I_3\mathbf{1}\boldsymbol{\omega}_3 \qquad \boldsymbol{\omega}_3=\begin{bmatrix}\omega_{13}\\ \omega_{23}\\ \omega_{33}\end{bmatrix}$$

Example: principal axes of a uniform cube

$$\frac{2Ma^2}{3}\omega_{13} - \frac{Ma^2}{4}\omega_{23} - \frac{Ma^2}{4}\omega_{33} = \frac{Ma^2}{6}\omega_{13} \qquad \frac{2\omega_{13}}{\omega_{33}} - \frac{\omega_{23}}{\omega_{33}} = 1$$

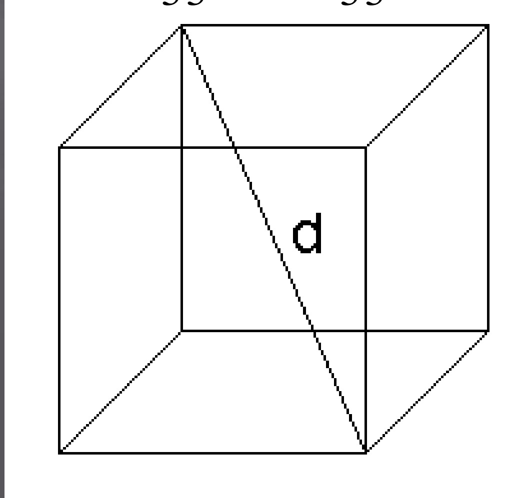
$$-\frac{Ma^2}{4}\omega_{13} + \frac{2Ma^2}{3}\omega_{23} - \frac{Ma^2}{4}\omega_{33} = \frac{Ma^2}{6}\omega_{23} \qquad -\frac{\omega_{13}}{\omega_{33}} + \frac{2\omega_{23}}{\omega_{33}} = 1$$

$$-\frac{Ma^2}{4}\omega_{13} - \frac{Ma^2}{4}\omega_{23} + \frac{2Ma^2}{3}\omega_{33} = \frac{Ma^2}{6}\omega_{33} \qquad \frac{\omega_{13}}{\omega_{33}} + \frac{\omega_{23}}{\omega_{33}} = 2$$

$$\omega_{13} = \omega_{23}$$

$$\omega_{13} = \omega_{33}$$

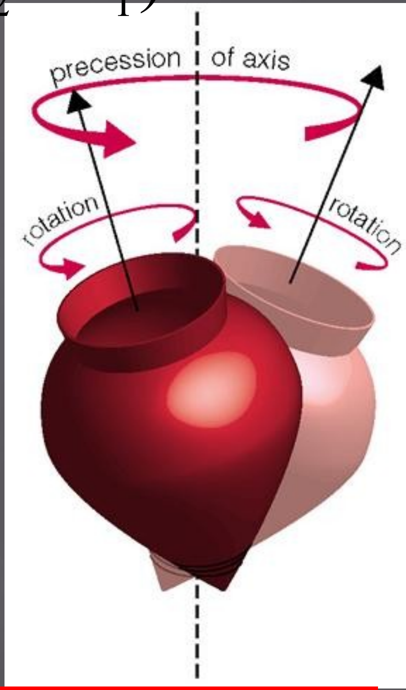
$$\boxed{\omega_{13} = \omega_{23} = \omega_{33}}$$



Free symmetrical top

$$I_i\dot{\omega}_i + \sum_{j,k=1}^{3}\varepsilon_{ijk}\omega_j\omega_k I_k = 0$$

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$$
$$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$$
$$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$$



- **For a free symmetrical top:**

$$I_1 = I_2 \neq I_3$$

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_1) = 0$$
$$I_1\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$$
$$I_3\dot{\omega}_3 = 0$$

$$\dot{\omega}_1 = \omega_2\frac{\omega_3(I_1 - I_3)}{I_1}$$

$$\dot{\omega}_2 = -\omega_1\frac{\omega_3(I_1 - I_3)}{I_1}$$

$$\omega_3 = const$$

$$\ddot{\omega}_1 = -\omega_1\left(\frac{\omega_3(I_1 - I_3)}{I_1}\right)^2 \equiv -\omega_1\alpha^2$$

$$\omega_1 = A\cos\alpha t$$
$$\omega_2 = A\sin\alpha t$$

Motion of non-free rigid bodies

• How to tackle rigid bodies that move in the presence of a **potential** or in an open system with generalized forces (**torques**)?

• Many Lagrangian problems of such types allow separation of the Lagrangians into two independent parts: the center-of-mass and the rotational

• For the non-Lagrangian (open) systems, we modify the equations of motion via introduction of generalized forces (torques) *N*:

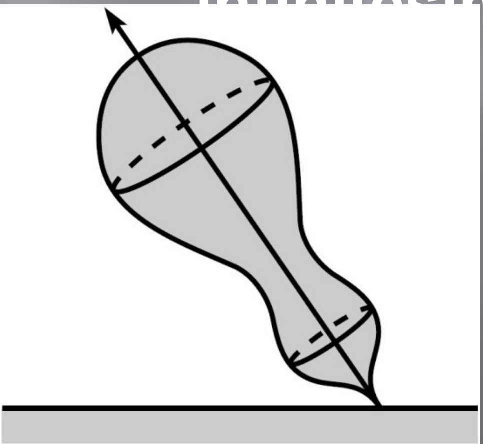
$$I_i\dot{\omega}_i + \sum_{j,k=1}^{3} \varepsilon_{ijk}\omega_j\omega_k I_k = N_i$$

Heavy symmetrical top with one point fixed

- For this problem, it is convenient to use the Euler angles as a set of independent variables

- Let us express the components of *ω* as functions of the Euler angles

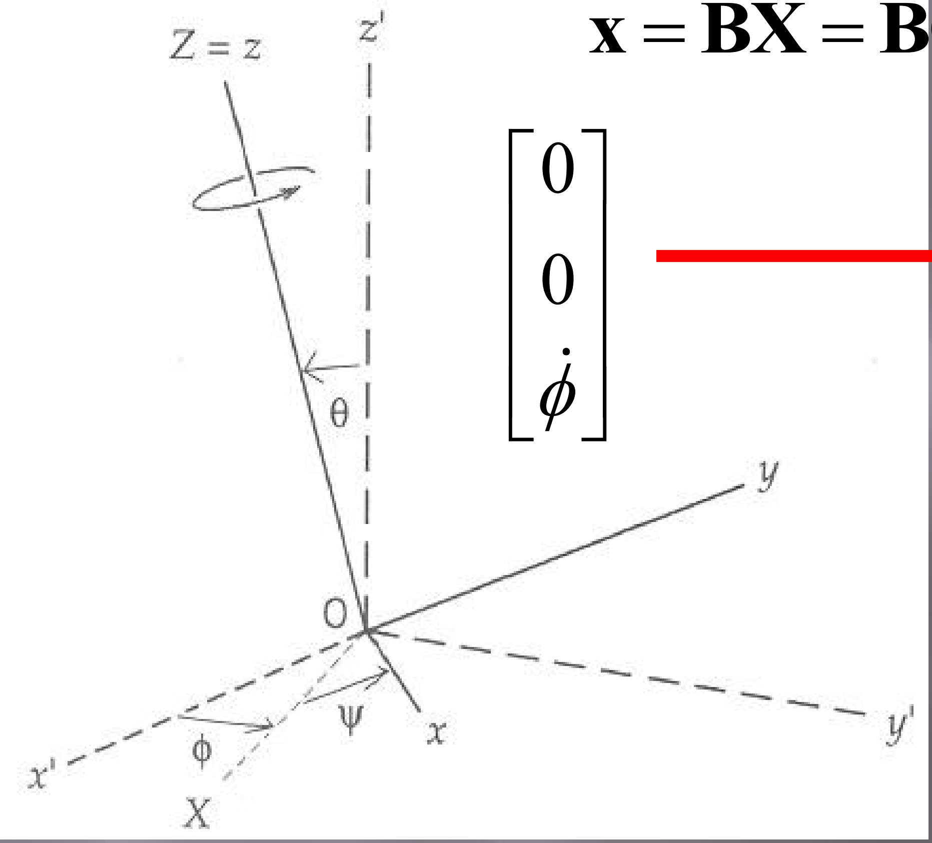
- The general infinitesimal rotation associated with *ω* can be considered as consisting of three successive infinitesimal rotations with angular velocities



$$\vec{\omega} = \vec{\omega}_\theta + \vec{\omega}_\phi + \vec{\omega}_\psi$$

$$\omega_\theta = \dot{\theta}; \omega_\phi = \dot{\phi}; \omega_\psi = \dot{\psi}$$

Heavy symmetrical top with one point fixed



$$\mathbf{x} = \mathbf{BX} = \mathbf{BCx''} = \mathbf{BCDx'} = \mathbf{Ax'}$$

$$\begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \rightarrow \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \mathbf{A} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$\dot{\phi}_x = \dot{\phi} \sin\psi \sin\theta$$

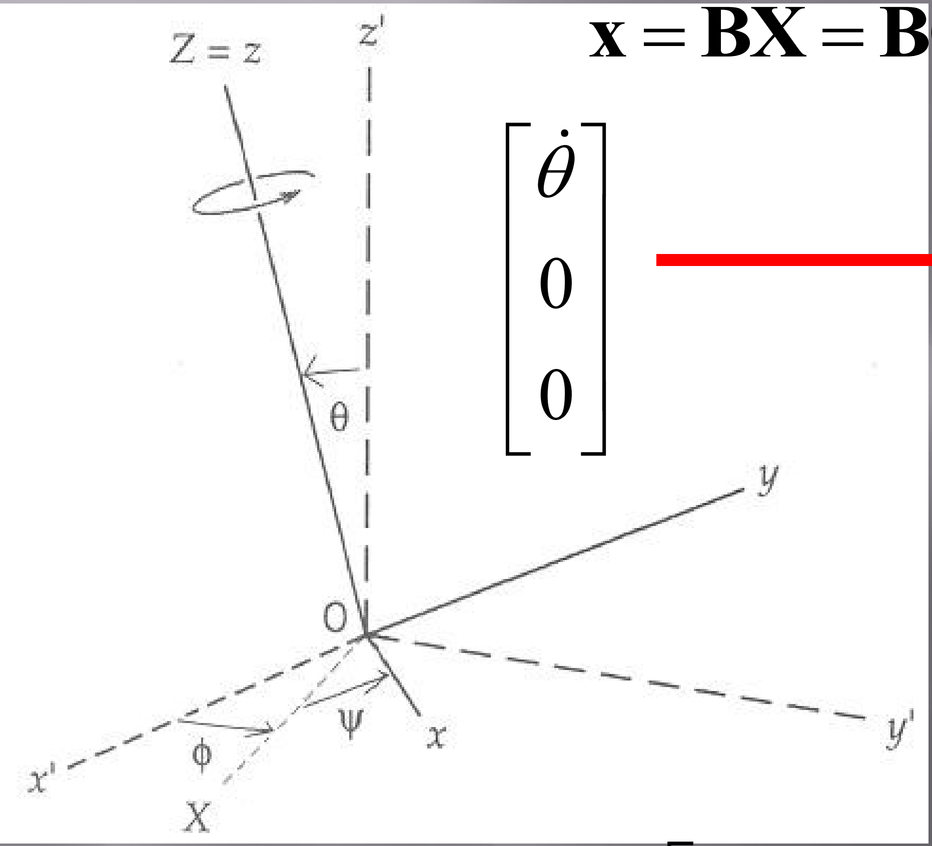
$$\dot{\phi}_y = \dot{\phi} \cos\psi \sin\theta$$

$$\dot{\phi}_z = \dot{\phi} \cos\theta$$

$$\mathbf{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

Heavy symmetrical top with one point fixed

$$\mathbf{x} = \mathbf{B}\mathbf{X} = \mathbf{B}\mathbf{C}\mathbf{x}'' = \mathbf{B}\mathbf{C}\mathbf{D}\mathbf{x}' = \mathbf{A}\mathbf{x}'$$



$$\begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \mathbf{B} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{\theta}_x = \dot{\theta}\cos\psi$$

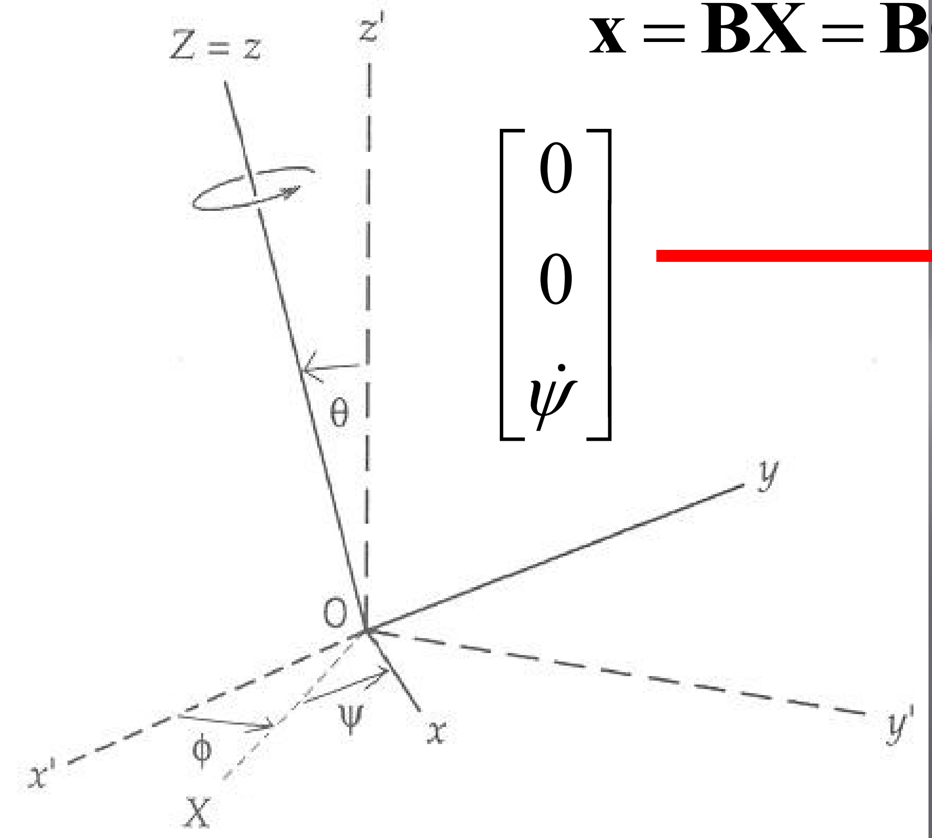
$$\dot{\theta}_y = -\dot{\theta}\sin\psi$$

$$\dot{\theta}_z = 0$$

$$\mathbf{B} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Heavy symmetrical top with one point fixed

$$\mathbf{x} = \mathbf{BX} = \mathbf{BCx''} = \mathbf{BCDx'} = \mathbf{Ax'}$$



$$\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\dot{\psi}_x = 0 \qquad \dot{\theta}_x = \dot{\theta}\cos\psi$$

$$\dot{\psi}_y = 0 \qquad \dot{\theta}_y = -\dot{\theta}\sin\psi$$

$$\dot{\psi}_z = \dot{\psi} \qquad \dot{\theta}_z = 0$$

$$\dot{\phi}_x = \dot{\phi}\sin\psi\sin\theta$$

$$\dot{\phi}_y = \dot{\phi}\cos\psi\sin\theta$$

$$\dot{\phi}_z = \dot{\phi}\cos\theta$$

Heavy symmetrical top with one point fixed

$$\vec{\omega} = \vec{\omega}_\theta + \vec{\omega}_\phi + \vec{\omega}_\psi$$

$$\dot{\psi}_x = 0 \qquad \dot{\theta}_x = \dot{\theta}\cos\psi$$

$$\dot{\psi}_y = 0 \qquad \dot{\theta}_y = -\dot{\theta}\sin\psi$$

$$\dot{\psi}_z = \dot{\psi} \qquad \dot{\theta}_z = 0$$

$$\vec{\omega} = \begin{bmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix}$$

$$\dot{\phi}_x = \dot{\phi}\sin\psi\sin\theta$$

$$\dot{\phi}_y = \dot{\phi}\cos\psi\sin\theta$$

$$\dot{\phi}_z = \dot{\phi}\cos\theta$$

Heavy symmetrical top with one point fixed

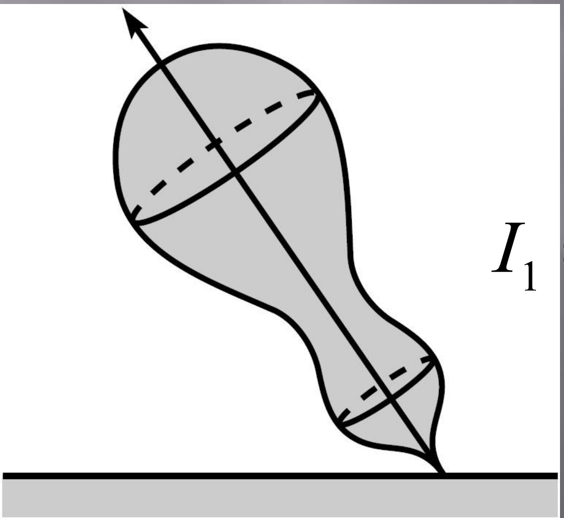
$$\sum_i m_i \vec{r}_i = M\vec{R}$$

- **The Lagrangian:** $L = T - V$

$$T = T_{Translation} + T_{Rotation} = \frac{I_1(\omega_1^2 + \omega_2^2)}{2} + \frac{I_3\omega_3^2}{2}$$

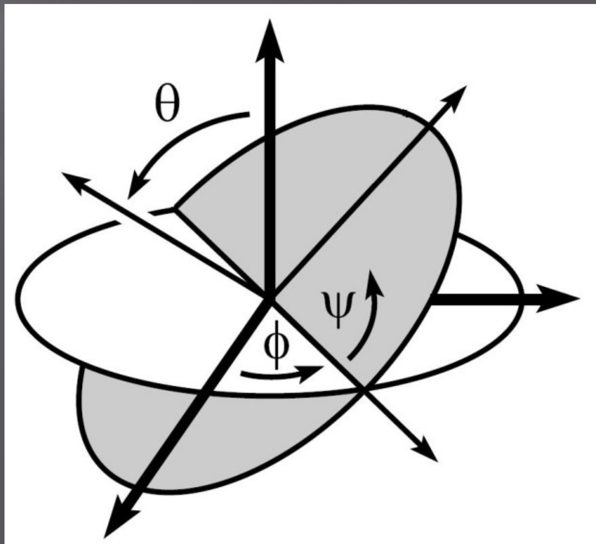
$$V = -\int \vec{r}\cdot\vec{g}\rho dV = -\vec{g}\cdot\int \vec{r}\rho dV = -\vec{g}\cdot\vec{R}M$$

- **Using the Euler angles**



$$V = gRM\cos\theta$$

$$I_1 = I_2$$



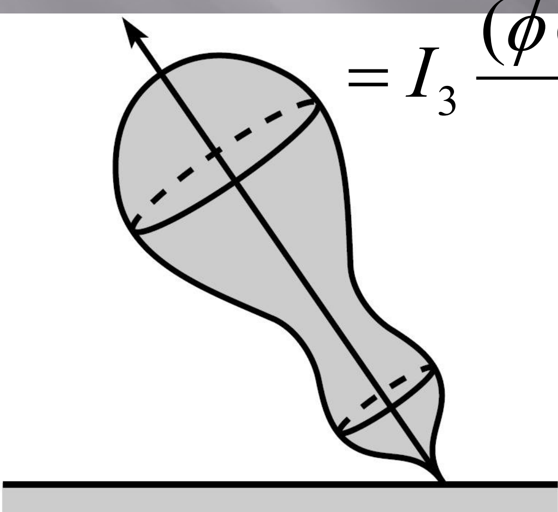
Heavy symmetrical top with one point fixed

$$\vec{\omega} = \begin{bmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix}$$

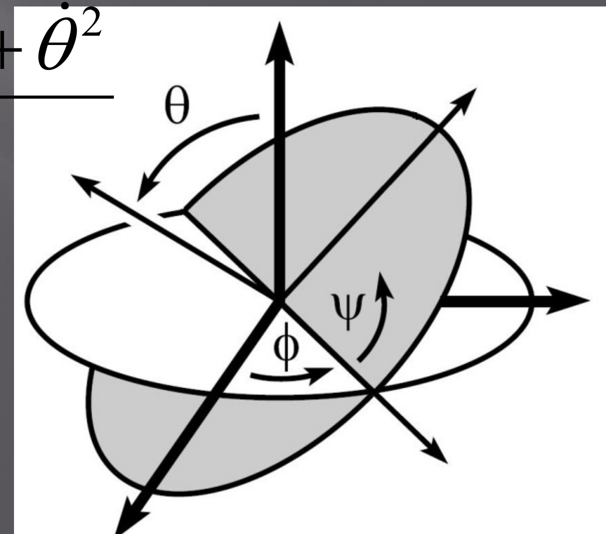
$$T = \frac{I_3 {\omega_3}^2}{2} + \frac{I_1({\omega_1}^2 + {\omega_2}^2)}{2} = \frac{I_3(\dot{\phi}\cos\theta + \dot{\psi})^2}{2}$$

$$+ I_1 \frac{(\dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)^2 + (\dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi)^2}{2}$$

$$= I_3 \frac{(\dot{\phi}\cos\theta + \dot{\psi})^2}{2} + I_1 \frac{\dot{\phi}^2\sin^2\theta + \dot{\theta}^2}{2}$$



$$V = gRM\cos\theta$$



Heavy symmetrical top with one point fixed

$$L = I_3 \frac{(\dot{\phi}\cos\theta + \dot{\psi})^2}{2} + I_1 \frac{\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2}{2} - gRM\cos\theta$$

- **The Lagrangian is cyclic in two coordinates**

$$\frac{\partial L}{\partial \phi} = 0; \frac{\partial L}{\partial \psi} = 0$$

- **Thus, we have two conserved generalized momenta**

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_3(\dot{\phi}\cos^2\theta + \dot{\psi}\cos\theta) + I_1(\dot{\phi}\sin^2\theta) = const \equiv I_1 b$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = const \equiv I_1 a$$

Heavy symmetrical top with one point fixed

$$L = I_3 \frac{(\dot{\phi}\cos\theta + \dot{\psi})^2}{2} + I_1 \frac{\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2}{2} - gRM\cos\theta$$

- The Lagrangian does not contain **time** explicitly $\frac{\partial L}{\partial t} = 0$
- Thus, the total energy of the system is conserved

$$E = I_3 \frac{(\dot{\phi}\cos\theta + \dot{\psi})^2}{2} + I_1 \frac{\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2}{2} + gRM\cos\theta = const$$

- To solve the problem completely, we need **three** additional quadratures
- We will look for them, using the conserved quantities

Heavy symmetrical top with one point fixed

$$I_3(\dot{\phi}\cos\theta + \dot{\psi}) = I_1 a \qquad I_3\dot{\psi} = I_1 a - I_3\dot{\phi}\cos\theta$$

$$I_3(\dot{\phi}\cos^2\theta + \dot{\psi}\cos\theta) + I_1(\dot{\phi}\sin^2\theta) = I_1 b$$

$$I_3(\dot{\phi}\cos^2\theta) + (I_1 a\cos\theta - I_3\dot{\phi}\cos^2\theta) + I_1(\dot{\phi}\sin^2\theta) = I_1 b$$

$$a\cos\theta + \dot{\phi}\sin^2\theta = b \qquad \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \equiv f_1(\theta)$$

$$\dot{\psi} = \frac{I_1}{I_3}a - f_1(\theta)\cos\theta \equiv f_2(\theta)$$

$$E = I_3\frac{(f_1(\theta)\cos\theta + f_2(\theta))^2}{2} + I_1\frac{f_1(\theta)^2\sin^2\theta + \dot{\theta}^2}{2} + gRM\cos\theta$$

- **One variable only: we can find all the quadratures!**

Heavy symmetrical top with one point fixed

$$E = I_3\frac{(f_1(\theta)\cos\theta + f_2(\theta))^2}{2} + I_1\frac{f_1(\theta)^2\sin^2\theta + \dot{\theta}^2}{2} + gRM\cos\theta$$

$$E = \frac{I_1\dot{\theta}^2}{2} + \frac{I_3(f_1(\theta)\cos\theta + f_2(\theta))^2}{2} + \frac{I_1 f_1(\theta)^2\sin^2\theta}{2} + RMg\cos\theta$$

- **We have an equivalent 1D problem with an effective potential!**

$$V_{eff}(\theta) = \frac{I_3(f_1(\theta)\cos\theta + f_2(\theta))^2}{2} + \frac{I_1 f_1(\theta)^2\sin^2\theta}{2} + RMg\cos\theta$$

$$= \frac{(I_1 a)^2}{2I_3} + \frac{I_1(b - a\cos\theta)^2}{2\sin^2\theta} + gRM\cos\theta$$

$$V_{eff}'(\theta) = \frac{I_1(b - a\cos\theta)^2}{2\sin^2\theta} + gRM\cos\theta$$

$$I_3(\dot{\phi}\cos\theta + \dot{\psi}) = I_1 a$$

Heavy symmetrical top with one point fixed

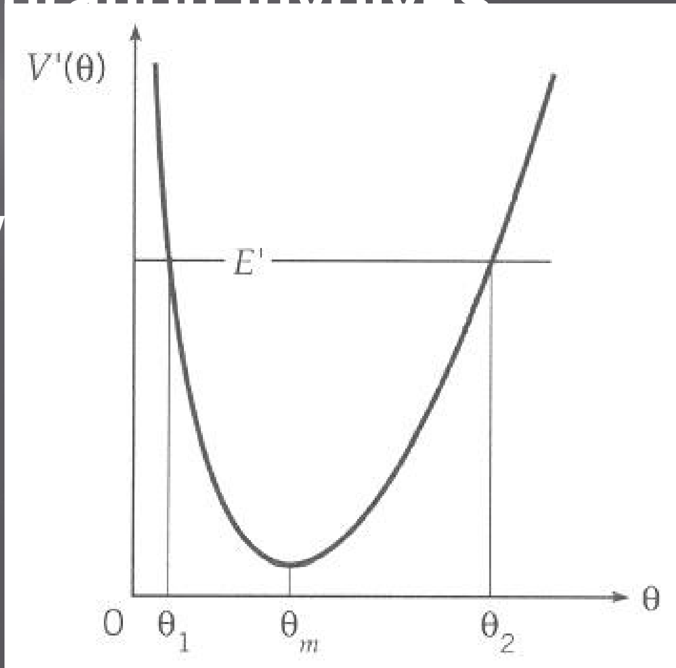
$$E' = \frac{I_1 \dot{\theta}^2}{2} + V_{eff}'(\theta)$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2[E' - V_{eff}'(\theta)]}{I_1}$$

$$t(\theta) = \int \frac{d\theta}{\sqrt{2/I_1[E' - V_{eff}'(\theta)]}}$$

- In the most general case, the integration involves **elliptic functions**

- Effective potential is a function w
  a minimum: motion in *θ* is bound
  two values



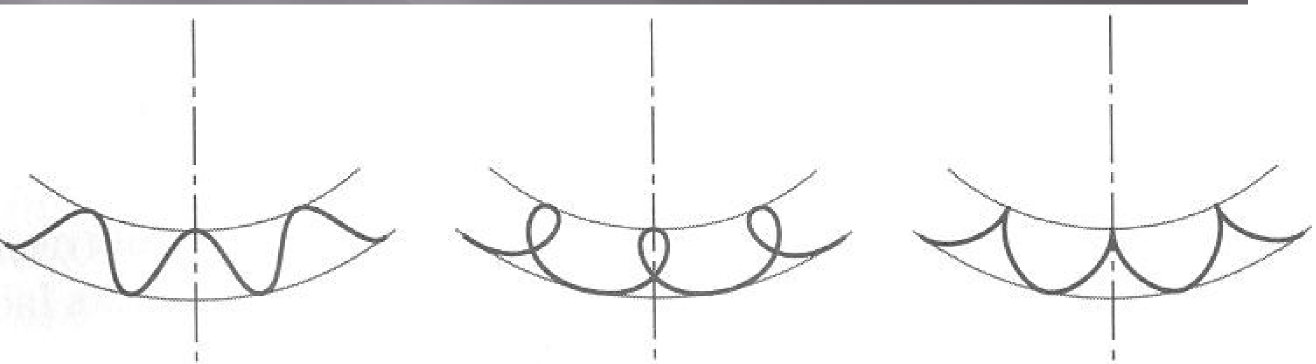
Heavy symmetrical top with one point fixed

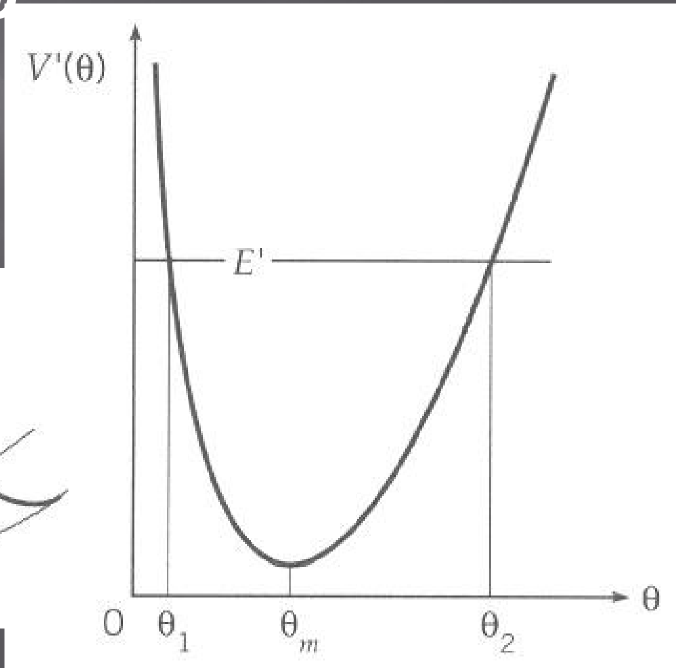
- When *θ* is at its minimum, we have a precession

- Otherwise, the top is bobbing: **nutation**

- The shape of the nutation trajectory depends on the behavior of the time derivative of *φ*

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$





Charged rigid body in an electromagnetic field

$$L = \sum_i \left(\frac{m_i(\dot{r}_{i,x}{}^2 + \dot{r}_{i,y}{}^2 + \dot{r}_{i,z}{}^2)}{2} - q_i \phi(\vec{r}_i) + q_i(\dot{\vec{r}}_i \cdot \vec{A}) \right)$$

- Let us consider the following vector potential (**C** – constant vector)

$$\vec{A} = \vec{C} \times \vec{r} \qquad A_i = (\vec{C} \times \vec{r})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} C_j r_k$$

- How is magnetic field related to vector **C**?

$$\vec{B} = \nabla \times \vec{A} \qquad B_n = (\nabla \times \vec{A})_n = \sum_{m,i=1}^{3} \varepsilon_{nmi} \frac{\partial A_i}{\partial r_m}$$

$$= \sum_{i,j,k,m=1}^{3} \varepsilon_{nmi} \varepsilon_{ijk} C_j \frac{\partial r_k}{\partial r_m} = \sum_{j,k,m=1}^{3} (\delta_{nj}\delta_{mk} - \delta_{nk}\delta_{mj}) C_j \frac{\partial r_k}{\partial r_m}$$

Charged rigid body in an electromagnetic field

$$B_n = \sum_{j,k,m=1}^{3} (\delta_{nj}\delta_{mk} - \delta_{nk}\delta_{mj}) C_j \frac{\partial r_k}{\partial r_m}$$

$$= \sum_{m=1}^{3} C_n \frac{\partial r_m}{\partial r_m} - \sum_{m=1}^{3} C_m \frac{\partial r_n}{\partial r_m} = 3C_n - \sum_{m=1}^{3} C_m \delta_{mn} = 2C_n = B_n$$

$$\vec{C} = \frac{\vec{B}}{2} \qquad \vec{A} = \frac{\vec{B} \times \vec{r}}{2}$$

- Constant magnetic field

$$L = \sum_i \left(\frac{m_i(\dot{r}_{i,x}{}^2 + \dot{r}_{i,y}{}^2 + \dot{r}_{i,z}{}^2)}{2} - q_i \phi(\vec{r}_i) + q_i \left(\dot{\vec{r}}_i \cdot \frac{\vec{B} \times \vec{r}_i}{2} \right) \right)$$

Charged rigid body in an electromagnetic field

$$L=\sum_i\left(\frac{m_i(\dot{r}_{i,x}{}^2+\dot{r}_{i,y}{}^2+\dot{r}_{i,z}{}^2)}{2}-q_i\phi(\vec{r}_i)+\dot{\vec{r}}_i\cdot\frac{q_i\vec{B}\times\vec{r}_i}{2}\right)$$

$$\sum_i\dot{\vec{r}}_i\cdot\frac{q_i\vec{B}\times\vec{r}_i}{2}=\sum_i\frac{q_i\vec{B}}{2m_i}\cdot(\vec{r}_i\times\dot{\vec{r}}_i m_i)$$

- Let us assume a **uniform** charge/mass ratio

$$=\frac{q\vec{B}}{2m}\cdot\sum_i(\vec{r}_i\times\dot{\vec{r}}_i m_i)=\frac{q\vec{B}}{2m}\cdot\vec{L}=\frac{(q/m)\widetilde{\mathbf{B}}\mathbf{L}}{2}$$

- Recall rotational kinetic energy

$$T_R=\frac{\widetilde{\boldsymbol{\omega}}\mathbf{L}}{2}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})$$

Radius of gyration

- FYI: radius of gyration is

$$R_0 = \sqrt{\frac{I}{M}}$$

$$I = MR_0^{\ 2}$$

$$T_R = \frac{I\omega^2}{2} = \frac{M(R_0\omega)^2}{2}$$