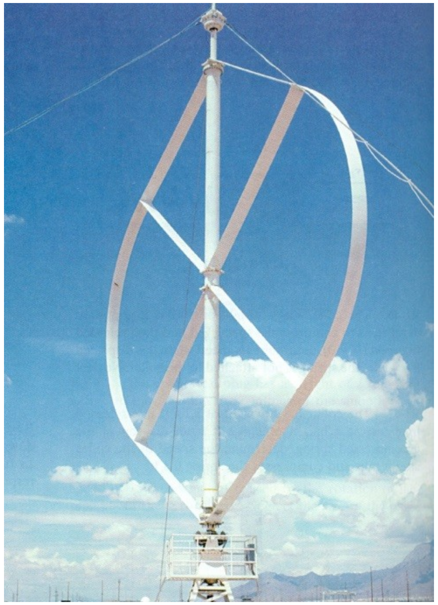
RIGID BODY MOTION: TRANSLATION & ROTATION (Sections 16.1-16.3)

Today's Objectives :

Students will be able to analyze the kinematics of a rigid body undergoing planar translation or rotation about a fixed axis.



In-Class Activities :

- Check homework, if any
- Reading quiz
- Applications
- Types of rigid-body motion
- Planar translation
- Rotation about a fixed axis
- Concept quiz
- Group problem solving
- Attention quiz

READING QUIZ

1. If a rigid body is in translation only, the velocity at points A and B on the rigid body ________ .

   A) are usually different
   B) are always the same
   C) depend on their position
   D) depend on their relative position

2. If a rigid body is rotating with a constant angular velocity about a fixed axis, the velocity vector at point P is _______.

   A) $\boldsymbol{\omega} \times \boldsymbol{r}_p$ B) $\boldsymbol{r}_p \times \boldsymbol{\omega}$

   C) $d\boldsymbol{r}_p/dt$ D) All of the above.

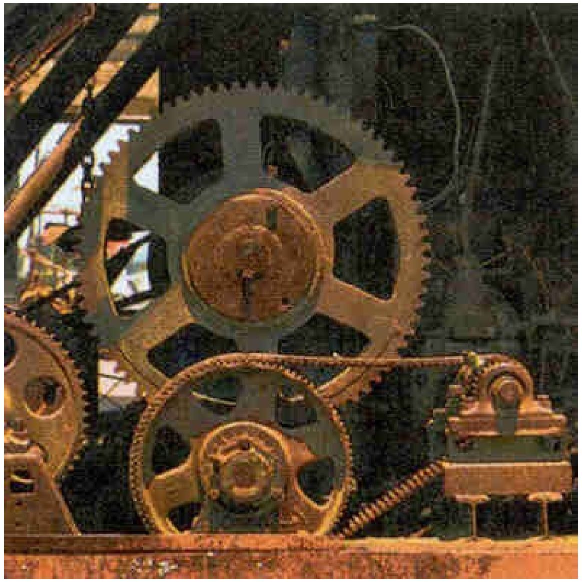
APPLICATIONS

Passengers on this amusement ride are subjected to curvilinear translation since the vehicle moves in a circular path but always remains upright.

If the angular motion of the rotating arms is known, how can we determine the velocity and acceleration experienced by the passengers?

Does each passenger feel the same acceleration?

APPLICATIONS (continued)



Gears, pulleys and cams, which rotate about fixed axes, are often used in machinery to generate motion and transmit forces. The angular motion of these components must be understood to properly design the system.

How can we relate the angular motions of contacting bodies that rotate about different fixed axes?

PLANAR KINEMATICS OF A RIGID BODY

There are cases where an object cannot be treated as a particle. In these cases the size or shape of the body must be considered. Also, rotation of the body about its center of mass requires a different approach.

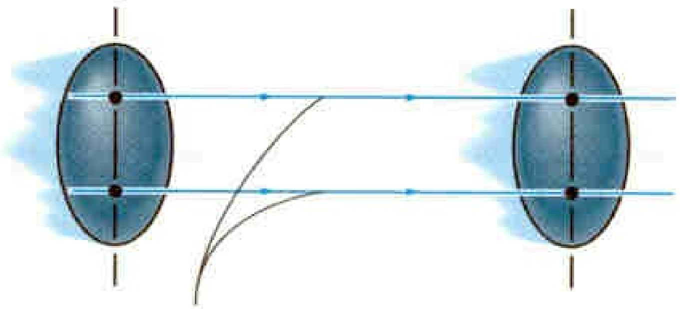
For example, in the design of gears, cams, and links in machinery or mechanisms, rotation of the body is an important aspect in the analysis of motion.

We will now start to study rigid body motion. The analysis will be limited to planar motion.

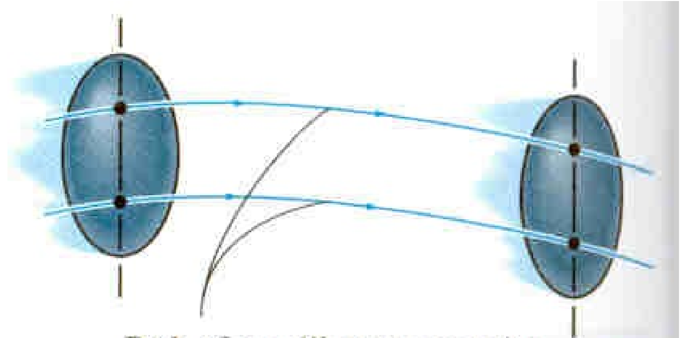
A body is said to undergo planar motion when all parts of the body move along paths equidistant from a fixed plane.

PLANAR RIGID BODY MOTION

There are three types of planar rigid body motion.



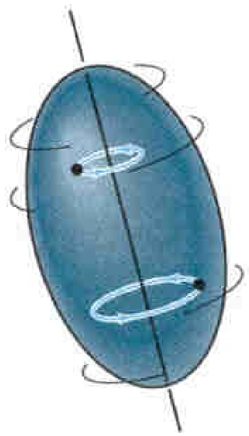
Path of rectilinear translation



Path of curvilinear translation

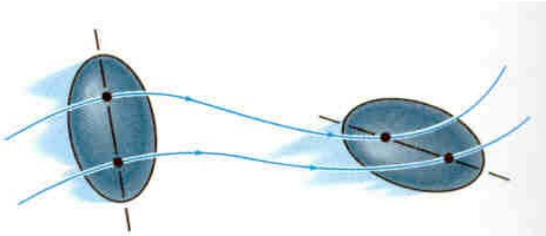
Translation: Translation occurs if every line segment on the body remains parallel to its original direction during the motion. When all points move along straight lines, the motion is called rectilinear translation. When the paths of motion are curved lines, the motion is called curvilinear translation.

PLANAR RIGID BODY MOTION (continued)



Rotation about a fixed axis

Rotation about a fixed axis. In this case, all the particles of the body, except those on the axis of rotation, move along circular paths in planes perpendicular to the axis of rotation.

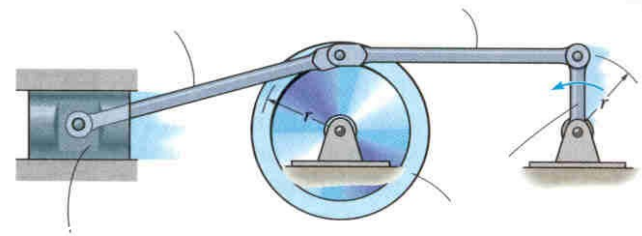


General plane motion

General plane motion. In this case, the body undergoes both translation and rotation. Translation occurs within a plane and rotation occurs about an axis perpendicular to this plane.

PLANAR RIGID BODY MOTION (continued)

An example of bodies undergoing the three types of motion is shown in this mechanism.

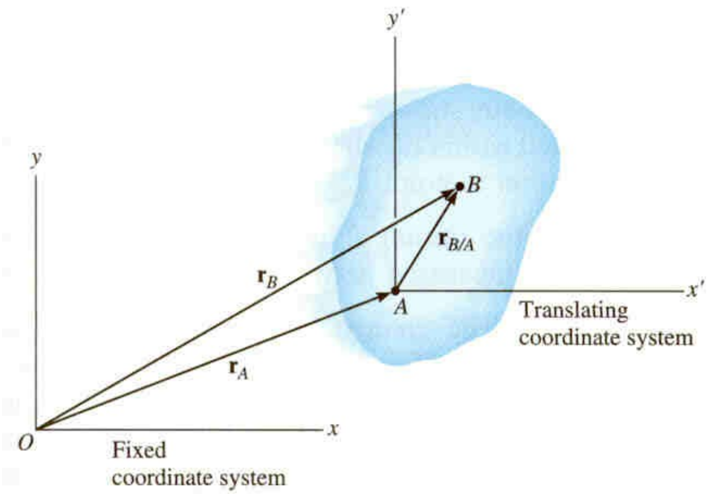


The wheel and crank (A and B) undergo rotation about a fixed axis. In this case, both axes of rotation are at the location of the pins and perpendicular to the plane of the figure.

The piston (C) undergoes rectilinear translation since it is constrained to slide in a straight line. The connecting rod (D) undergoes curvilinear translation, since it will remain horizontal as it moves along a circular path.

The connecting rod (E) undergoes general plane motion, as it will both translate and rotate.

RIGID-BODY MOTION: TRANSLATION



The positions of two points A and B on a translating body can be related by

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$

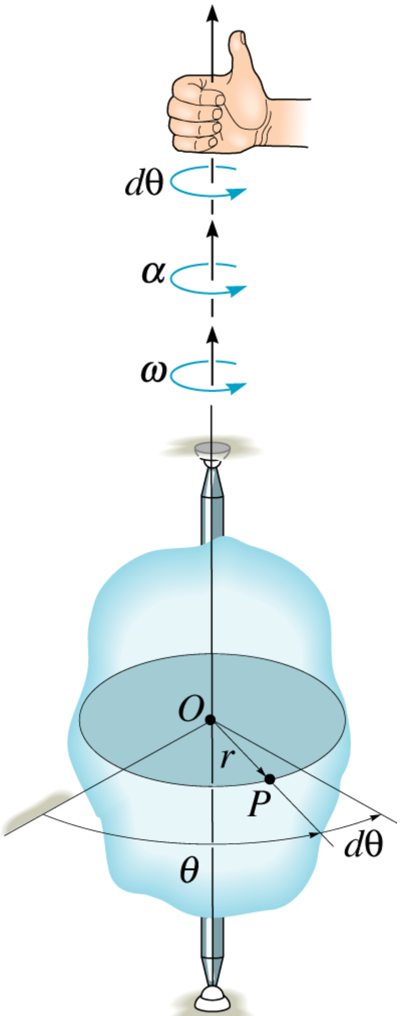
where $\boldsymbol{r}_A$ & $\boldsymbol{r}_B$ are the absolute position vectors defined from the fixed x-y coordinate system, and $\boldsymbol{r}_{B/A}$ is the relative-position vector between B and A.

The velocity at B is $\boldsymbol{v}_B = \boldsymbol{v}_A + d\boldsymbol{r}_{B/A}/dt$.

Now $d\boldsymbol{r}_{B/A}/dt = \boldsymbol{0}$ since $\boldsymbol{r}_{B/A}$ is constant. So, $\boldsymbol{v}_B = \boldsymbol{v}_A$, and by following similar logic, $\boldsymbol{a}_B = \boldsymbol{a}_A$.

Note, all points in a rigid body subjected to translation move with the same velocity and acceleration.

RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS



When a body rotates about a fixed axis, any point P in the body travels along a circular path. The angular position of P is defined by θ.

The change in angular position, $d\theta$, is called the angular displacement, with units of either radians or revolutions. They are related by

$$1 \text{ revolution} = 2\pi \text{ radians}$$

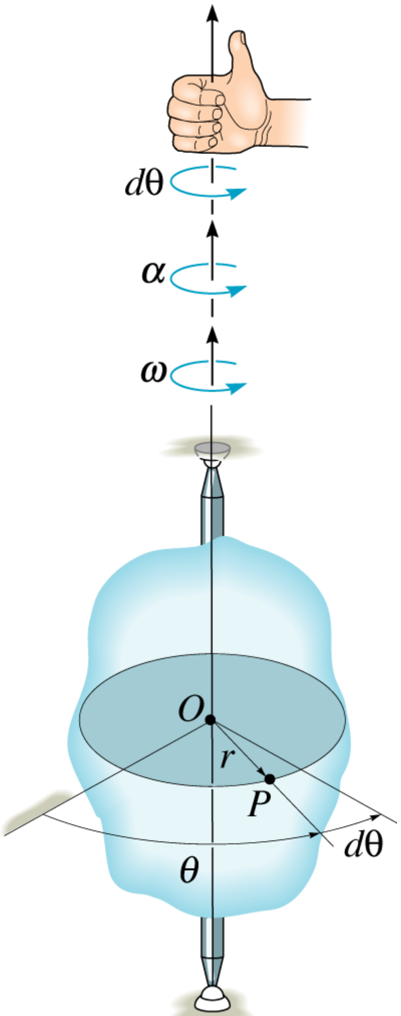
Angular velocity, ω, is obtained by taking the time derivative of angular displacement:

$$\omega = d\theta/dt \text{ (rad/s)} \; +$$

Similarly, angular acceleration is

$$\alpha = d^2\theta/dt^2 = d\omega/dt \quad \text{or } \alpha = \omega(d\omega/d\theta) \; + \quad \text{rad/s}^2$$

RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS (continued)



If the angular acceleration of the body is constant, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_C$, the equations for angular velocity and acceleration can be integrated to yield the set of algebraic equations below.

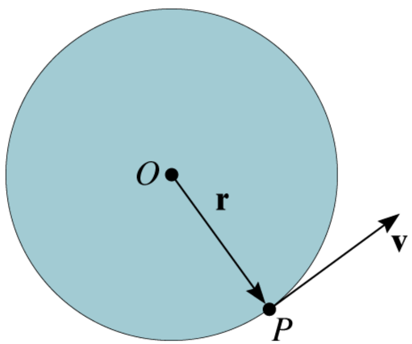
$$\omega = \omega_O + \alpha_C t$$

$$\theta = \theta_O + \omega_O t + 0.5\alpha_C t^2$$

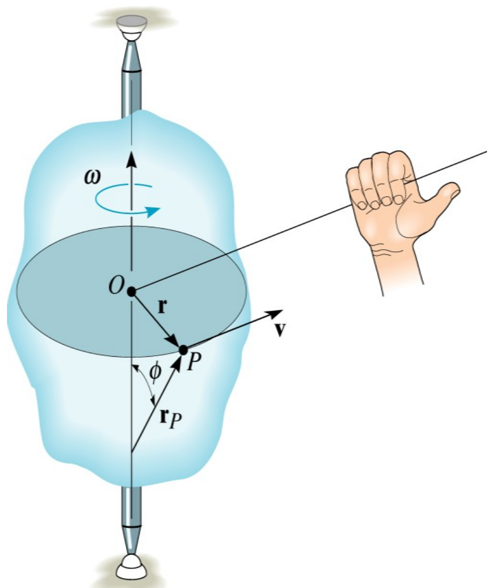
$$\omega^2 = (\omega_O)^2 + 2\alpha_C\,(\theta - \theta_O)$$

θ_O and ω_O are the initial values of the body's angular position and angular velocity. Note these equations are very similar to the constant acceleration relations developed for the rectilinear motion of a particle.

RIGID-BODY ROTATION: VELOCITY OF POINT P



The magnitude of the velocity of P is equal to ωr (the text provides the derivation). The velocity's direction is tangent to the circular path of P.

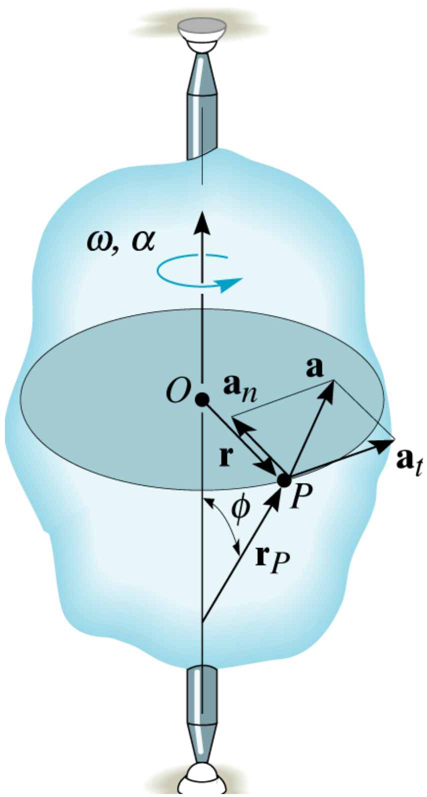


In the vector formulation, the magnitude and direction of $\boldsymbol{v}$ can be determined from the cross product of $\boldsymbol{\omega}$ and $\boldsymbol{r}_p$. Here $\boldsymbol{r}_p$ is a vector from any point on the axis of rotation to P.

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_p = \boldsymbol{\omega} \times \boldsymbol{r}$$

The direction of $\boldsymbol{v}$ is determined by the right-hand rule.

RIGID-BODY ROTATION: ACCELERATION OF POINT P

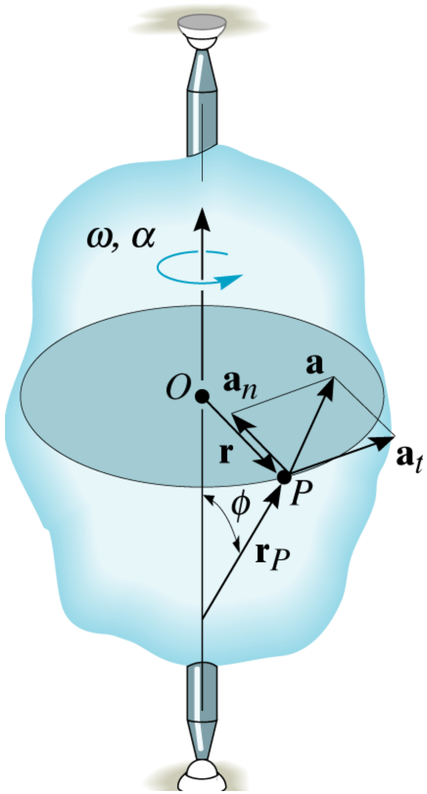


The acceleration of P is expressed in terms of its normal (***a***$_n$) and tangential (***a***$_t$) components. In scalar form, these are $a_t = \alpha\, r$ and $a_n = \omega^2 r$.

The tangential component, ***a***$_t$, represents the time rate of change in the velocity's magnitude. It is directed tangent to the path of motion.

The normal component, ***a***$_n$, represents the time rate of change in the velocity's direction. It is directed toward the center of the circular path.

RIGID-BODY ROTATION: ACCELERATION OF POINT P
(continued)



Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity.

$$\boldsymbol{a} = d\boldsymbol{v}/dt = d\boldsymbol{\omega}/dt \times \boldsymbol{r}_P + \boldsymbol{\omega} \times d\boldsymbol{r}_P/dt$$

$$= \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P)$$

It can be shown that this equation reduces to

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$$

The magnitude of the acceleration vector is $a = \sqrt{(a_t)^2 + (a_n)^2}$

ROTATION ABOUT A FIXED AXIS: PROCEDURE

- Establish a sign convention along the axis of rotation.

- If a relationship is known between any two of the variables (α, ω, θ, or t), the other variables can be determined from the equations: $\omega = d\theta/dt$ $\alpha = d\omega/dt$ $\alpha\, d\theta = \omega\, d\omega$

- If α is constant, use the equations for constant angular acceleration.

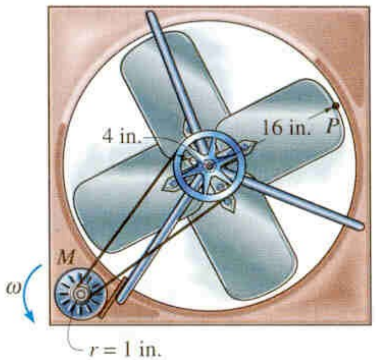
- To determine the motion of a point, the scalar equations $v = \omega r$, $a_t = \alpha r$, $a_n = \omega^2 r$, and $a = \sqrt{(a_t)^2 + (a_n)^2}$ can be used.

- Alternatively, the vector form of the equations can be used (with ***i***, ***j***, ***k*** components).

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_P = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{a} = \boldsymbol{a}_t + \boldsymbol{a}_n = \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P) = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r}$$

EXAMPLE



Given: The motor M begins rotating at $\omega = 4(1 - e^{-t})$ rad/s, where t is in seconds. The radii of the motor, fan pulleys, and fan blades are 1 in, 4 in, and 16 in, respectively.

Find: The magnitudes of the velocity and acceleration at point P on the fan blade when t = 0.5 s.

Plan:
1) Determine the angular velocity and acceleration of the motor using kinematics of angular motion.
2) Assuming the belt does not slip, the angular velocity and acceleration of the fan are related to the motor's values by the belt.
3) The magnitudes of the velocity and acceleration of point P can be determined from the scalar equations of motion for a point on a rotating body.

EXAMPLE (continued)

Solution:

1) Since the angular velocity is given as a function of time, $\omega_m = 4(1 - e^{-t})$, the angular acceleration can be found by differentiation.

$$\alpha_m = d\omega_m/dt = 4e^{-t} \ \text{rad/s}^2$$

When t = 0.5 s,

$\omega_m = 4(1 - e^{-0.5}) = 1.5739$ rad/s, $\alpha_m = 4e^{-0.5} = 2.4261$ rad/s^2

2) Since the belt does not slip (and is assumed inextensible), it must have the same speed and tangential component of acceleration at all points. Thus the pulleys must have the same speed and tangential acceleration at their contact points with the belt. Therefore, the angular velocities of the motor (ω_m) and fan (ω_f) are related as

$$v = \omega_m r_m = \omega_f r_f \Rightarrow (1.5739)(1) = \omega_f(4) \Rightarrow \omega_f = 0.3935 \text{ rad/s}$$

EXAMPLE (continued)

3) Similarly, the tangential accelerations are related as

$$a_t = \alpha_m r_m = \alpha_f r_f \Rightarrow (2.4261)(1) = \alpha_f(4) \Rightarrow \alpha_f = 0.6065 \text{ rad/s}^2$$

4) The speed of point P on the the fan, at a radius of 16 in, is now determined as

$$v_P = \omega_f r_P = (0.3935)(16) = 6.30 \text{ in/s}$$

The normal and tangential components of acceleration of point P are calculated as

$$a_n = (\omega_f)^2 r_P = (0.3935)^2 (16) = 2.477 \text{ in/s}^2$$

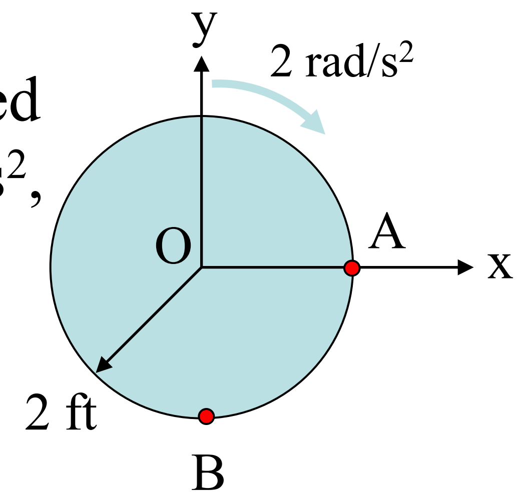
$$a_t = \alpha_f r_P = (0.6065)(16) = 9.704 \text{ in/s}^2$$

The magnitude of the acceleration of P can be determined by

$$a_P = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{(2.477)^2 + (9.704)^2} = 10.0 \text{ in/s}^2$$

CONCEPT QUIZ

1. A disk is rotating at 4 rad/s. If it is subjected to a constant angular acceleration of 2 rad/s^2, determine the acceleration at B.



A) $(4\,\boldsymbol{i} + 32\,\boldsymbol{j})$ ft/s^2 B) $(4\,\boldsymbol{i} - 32\,\boldsymbol{j})$ ft/s^2

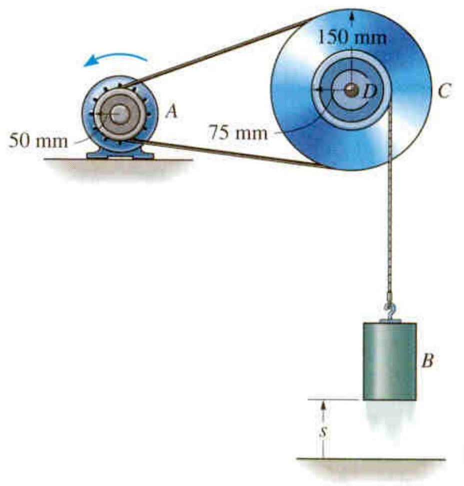
C) $(-4\,\boldsymbol{i} + 32\,\boldsymbol{j})$ ft/s^2 D) $(-4\,\boldsymbol{i} - 32\,\boldsymbol{j})$ ft/s^2

2. A Frisbee is thrown and curves to the right. It is experiencing

A) rectilinear translation. B) curvilinear translation.

C) pure rotation. D) general plane motion.

GROUP PROBLEM SOLVING



Given: Starting from rest when s = 0, pulley A (r_A = 50 mm) is given a constant angular acceleration, $\alpha_A = 6\ \text{rad/s}^2$. Pulley C ($r_C$ = 150 mm) has an inner hub D (r_D = 75 mm) which is fixed to C and turns with it.

Find: The speed of block B when it has risen s = 6 m.

Plan:
1) The angular acceleration of pulley C (and hub D) can be related to α_A if it is assumed the belt is inextensible and does not slip.
2) The acceleration of block B can be determined by using the equations for motion of a point on a rotating body.
3) The velocity of B can be found by using the constant acceleration equations.

GROUP PROBLEM SOLVING (continued)

Solution:

1) Assuming the belt is inextensible and does not slip, it will have the same speed and tangential component of acceleration as it passes over the two pulleys (A and C). Thus,

$$a_t = \alpha_A r_A = \alpha_C r_C \quad \Rightarrow \quad (6)(50) = \alpha_C(150) \quad \Rightarrow \quad \alpha_C = 2\ \text{rad/s}^2$$

Since C and D turn together, $\alpha_D = \alpha_C = 2\ \text{rad/s}^2$

2) Assuming the cord attached to block B is inextensible and does not slip, the speed and acceleration of B will be the same as the speed and tangential component of acceleration along the outer rim of hub D:

$$a_B = (a_t)_D = \alpha_D r_D = (2)(0.075) = 0.15\ \text{m/s}^2 \uparrow$$

GROUP PROBLEM SOLVING (continued)

3) Since α_A is constant, α_D and a_B will be constant. The constant acceleration equation for rectilinear motion can be used to determine the speed of block B when s = 6 m ($s_o = v_o = 0$):

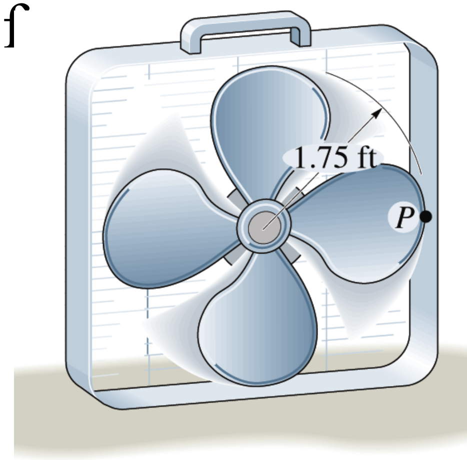
$$(v_B)^2 = (v_o)^2 + 2a_B(s - s_o) \quad + \uparrow$$

$$(v_B)^2 = 0 + 2(0.15)(6 - 0)$$

$$v_B = 1.34 \text{ m/s} \uparrow$$

ATTENTION QUIZ

1. The fan blades suddenly experience an angular acceleration of 2 rad/s^2. If the blades are rotating with an initial angular velocity of 4 rad/s, determine the speed of point P when the blades have turned 2 revolutions (when $\omega = 8.14$ rad/s).



   A) 14.2 ft/s B) 17.7 ft/s

   C) 23.1 ft/s D) 26.7 ft/s

2. Determine the magnitude of the acceleration at P when the blades have turned the 2 revolutions.

   A) 0 ft/s^2 B) 3.5 ft/s^2

   C) 115.95 ft/s^2 D) 116 ft/s^2