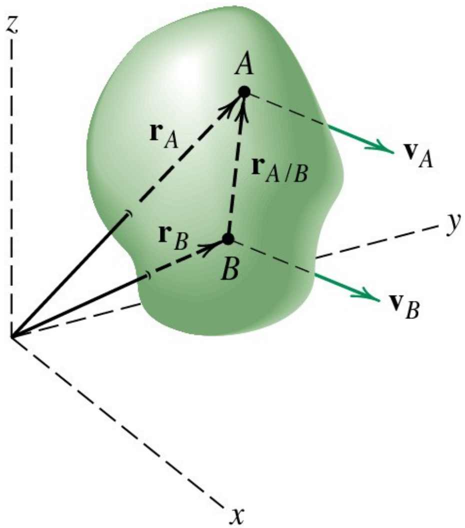
THREE-DIMENSIONAL KINEMATICS OF RIGID BODIES

1. TRANSLATION



$$\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$$

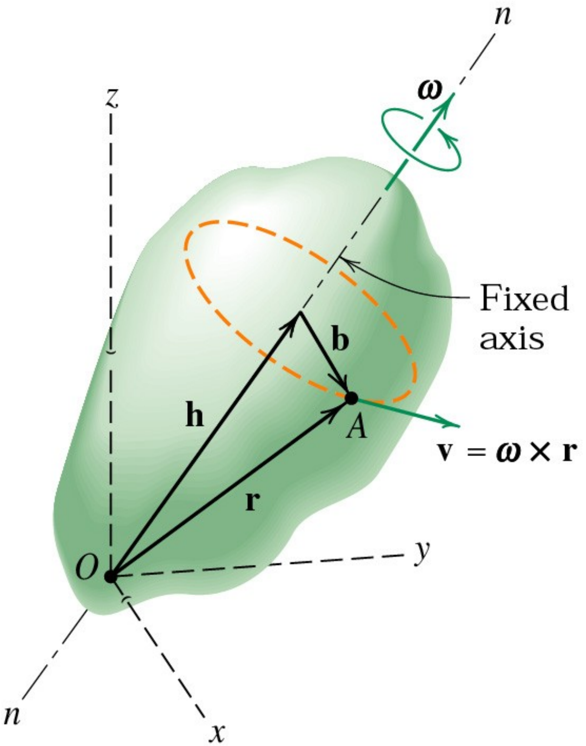
$$\vec{v}_A = \vec{v}_B$$

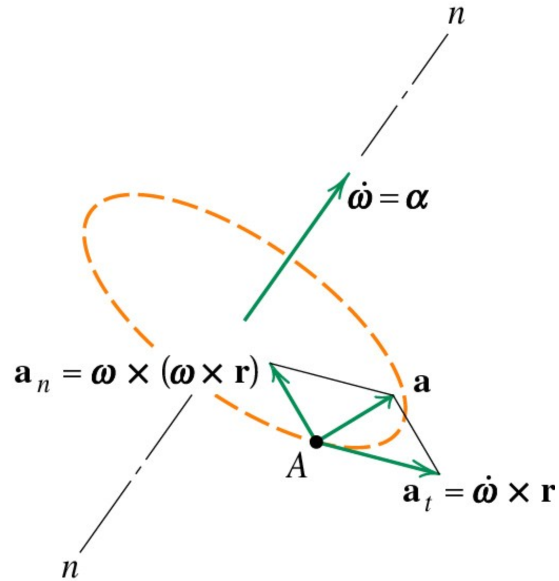
$$\vec{a}_A = \vec{a}_B$$

$\vec{r}_{A/B}$ remains constant and therefore its time derivative is zero.

Thus, all points in the body have the same velocity and the same acceleration.

2. FIXED-AXIS ROTATION





Any point such as A which is not on the axis moves in a circular arc in a plane normal to the axis and has a velocity and an acceleration

$$\vec{v} = \vec{\omega} \times \vec{r}$$

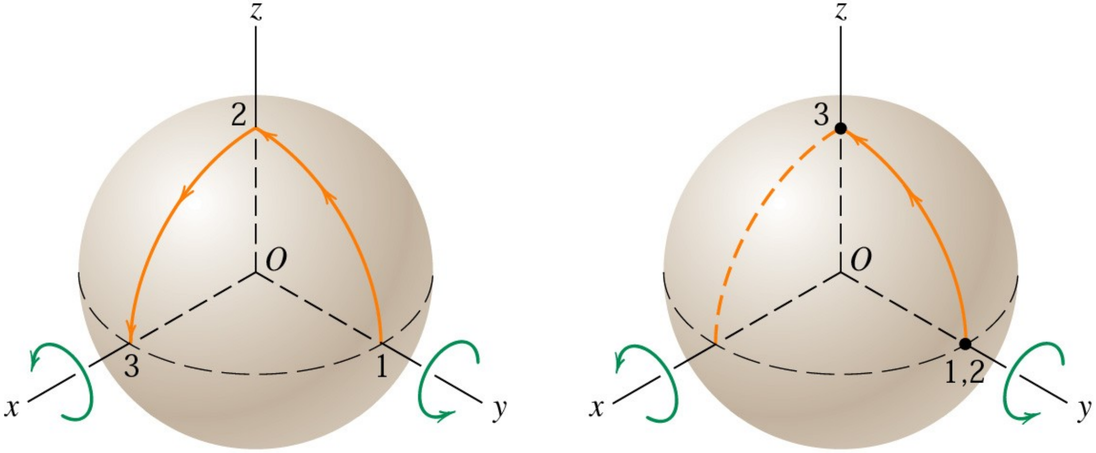
$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)$$

3. ROTATION ABOUT A FIXED POINT

When a body rotates about a fixed point, the angular velocity vector no longer remains fixed in direction, and this change calls for a more general concept of rotation.

Rotation and Proper Vectors

Consider a solid sphere which is cut from a rigid body confined to rotate about the fixed point P. The *x-y-z* axis here are taken as fixed in space and do not rotate with the body.



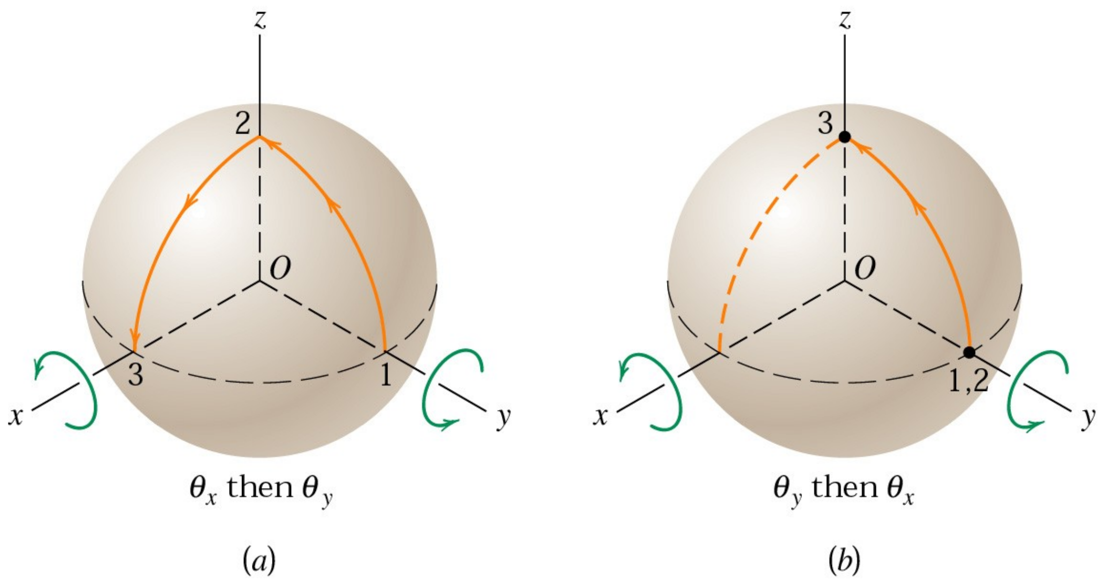
θ_x then θ_y

(*a*)

θ_y then θ_x

(*b*)

In part (a) of the figure, two successive 90° rotations of the sphere about, first, the x-axis and second, the y axis result in the motion of a point which is initially on the y-axis in position 1, to positions 2 and 3, successively.

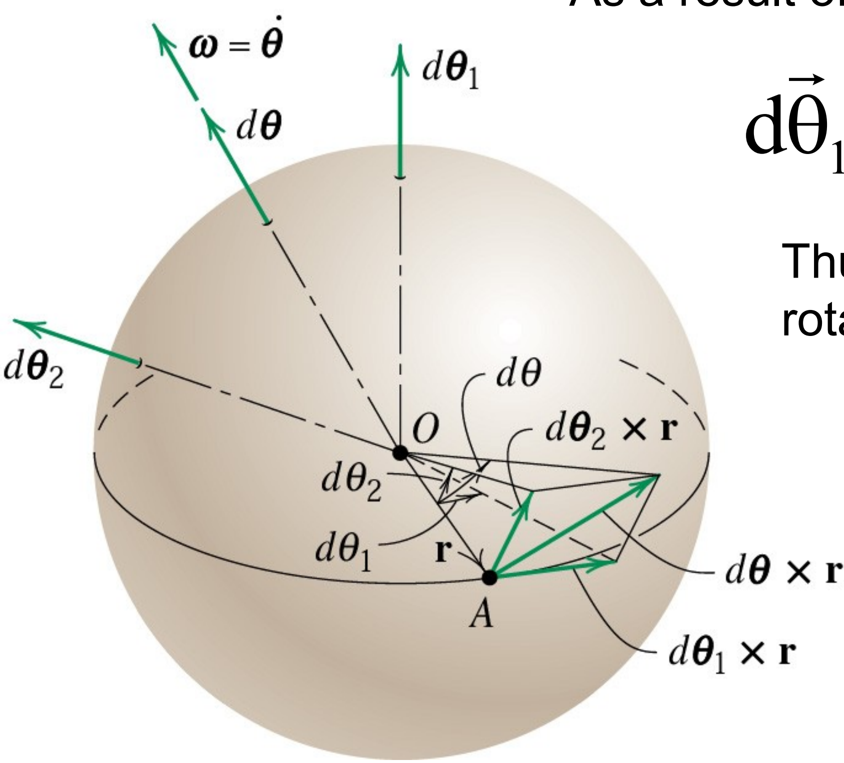


On the other hand, if the order of the rotations is reversed, the point undergoes no motion during the *y*-rotation but moves to point 3 during the 90° rotation about *x*-axis. Thus, the two cases do not produce the same final position, and it is evident from this one special example that finite rotations do not generally obey the parallelogram law of vector addition and are not commutative. $(a+b \neq b+a)$

Thus, finite rotations may not be treated as proper vectors.

Infinitesimal rotations do obey the parallelogram law of vector addition.

Infinitesimal rotations are $d\theta_1$ and $d\theta_1$



As a result of $d\theta_1$ and $d\theta_2$, point A has a displacement

$$d\vec{\theta}_1 \times \vec{r} + d\vec{\theta}_2 \times \vec{r} = \left(d\vec{\theta}_1 + d\vec{\theta}_2\right) \times \vec{r}$$

Thus, two rotations are equivalent to the single rotation

$$d\vec{\theta} = d\vec{\theta}_1 + d\vec{\theta}_2$$

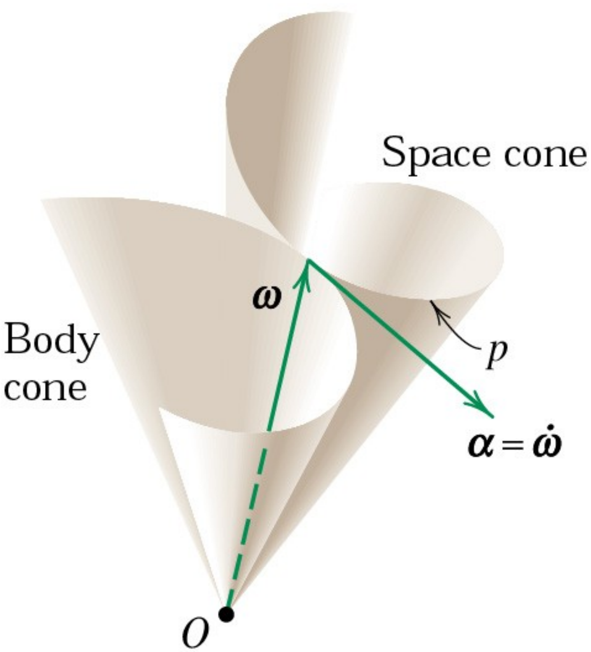
$$\dot{\vec{\theta}}_1 = \vec{\omega}_1 \qquad \dot{\vec{\theta}}_2 = \vec{\omega}_2$$

$$\vec{\omega} = \dot{\vec{\theta}} = \vec{\omega}_1 + \vec{\omega}_2$$

Angular Acceleration

The angular acceleration α of rigid body in three-dimensional motion is the time derivative of its angular velocity

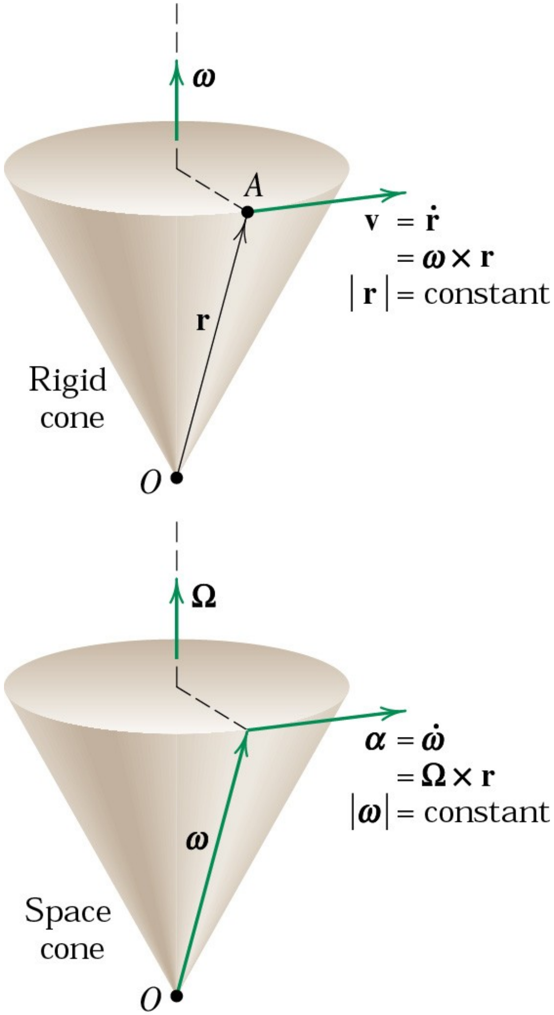
$$\vec{\alpha} = \dot{\vec{\omega}}$$



In contrast to the case of rotation in a single plane where the scalar a measures only the change in magnitude of the angular velocity, in three-dimensional motion the vector α reflects the change in direction as well as its change in magnitude.

Thus in the figure where the tip of the angular velocity vector ω follows the space curve p and changes in both magnitude and direction, the angular acceleration a becomes α vector tangent to this curve in the direction of the change in ω.

When the magnitude of ω remains constant, the angular acceleration α is normal to ω. For this case, if we let Ω stand for the angular velocity with the vector ω itself rotates (precesses) as it forms the space cone, the angular acceleration may be written



$$\vec{\alpha} = \vec{\Omega} \times \vec{\omega}$$

The upper part of the figure relates the velocity of point A on a rigid body to its position vector from O and the angular velocity of the body. The vectors α, ω, and Ω in the lower figure bear exactly the same relationship to each other as do vectors **v**, **r** and $\boldsymbol{\omega}$ in the upper figure.

The one difference between the case of rotation about a fixed axis and rotation about a fixed point lies in the fact that for rotation about a fixed point, the angular acceleration

$$\vec{\alpha} = \dot{\vec{\omega}}$$

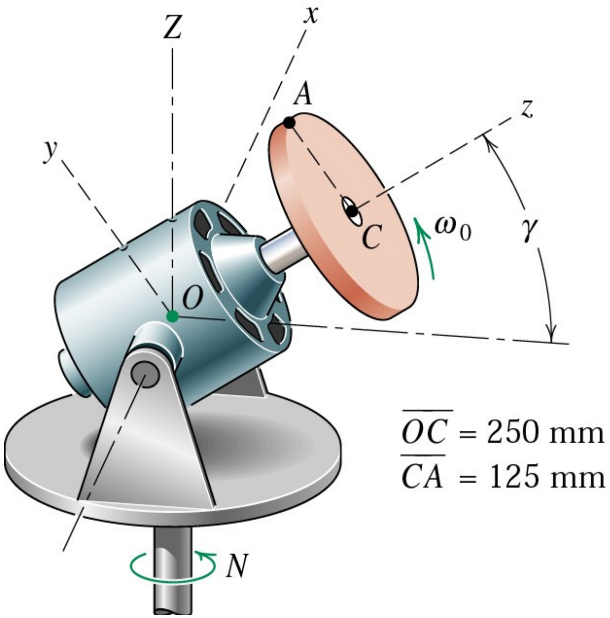
will have a component normal to ω due to the change in direction of ω.

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)$$

Although any point on the rotation axis *n-n* momentarily will have zero velocity, it will not have zero acceleration as long as ω is changing its direction.

Example

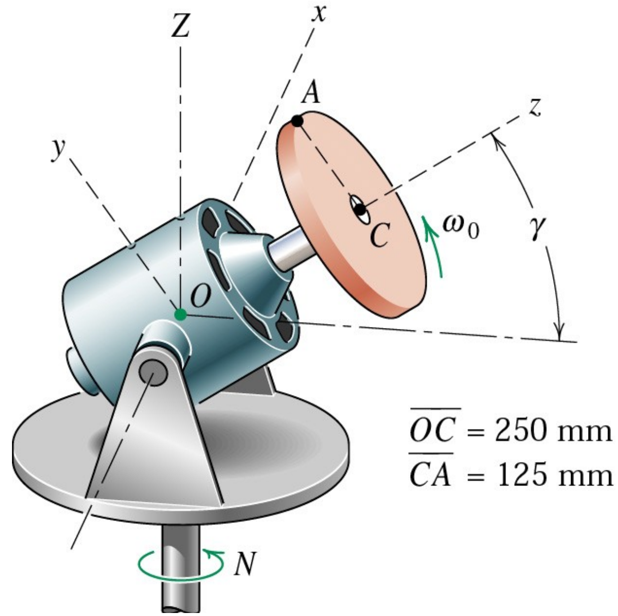


The electric motor with an attached disk is running at a constant low speed of 120 rev/min in the direction shown. Its housing and mounting base are initially rest. The entire assembly is next set in rotation about the vertical Z-axis at the constant rate N=60 rev/min with a fixed angle γ of 30°. Determine **(a)** the angular velocity and angular acceleration of the disk, **(b)** the space and body cone, and **(c)** the velocity and acceleration of point A at the top of the disk for the instant shown.

The axes x-y-z with unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are attached to the motor frame.

$$\vec{K} = \vec{j}\cos\gamma + \vec{k}\sin\gamma$$

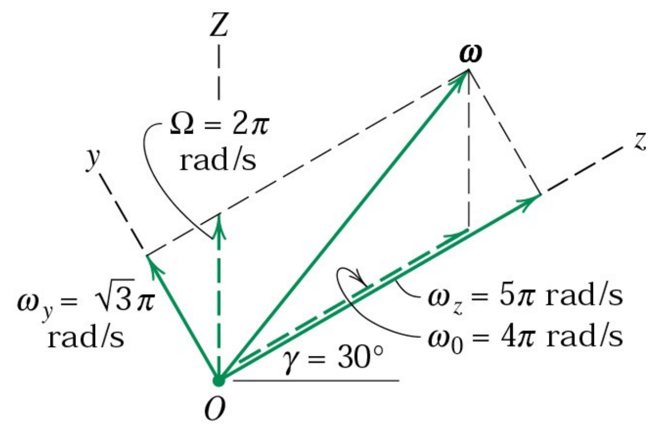
(a) the angular velocity and angular acceleration of the disk



The rotor and disk have two components of angular velocity : $\omega_o=120(2\pi)/60=4\pi$ rad/s about the z-axis and $\Omega= 60(2\pi)/60=2\pi$ rad/s about the Z-axis.

$$\vec{\omega} = \vec{\omega}_o + \vec{\Omega} = \omega_o \vec{k} + \Omega \vec{K}$$

$$\vec{\omega} = \omega_o \vec{k} + \Omega\left(\vec{j}\cos\gamma + \vec{k}\sin\gamma\right) = \pi\left(\sqrt{3}\,\vec{j} + 5\vec{k}\right)$$



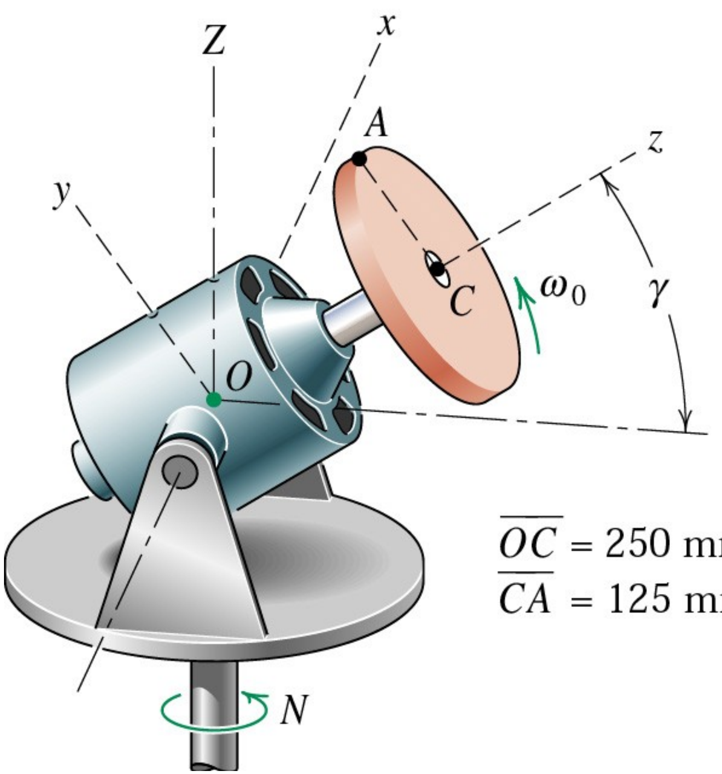
The angular acceleration of the disk

$$\vec{\alpha} = \dot{\vec{\omega}} = \vec{\Omega} \times \vec{\omega}$$

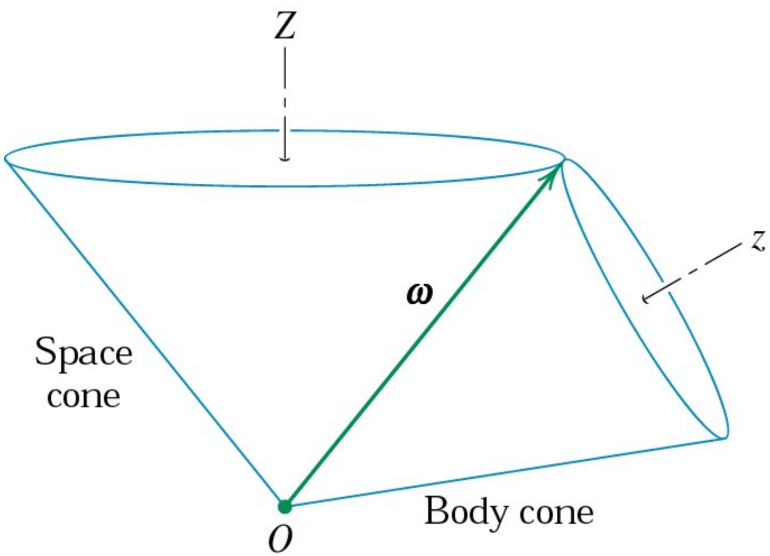
$$\vec{\alpha} = \Omega\left(\vec{j}\cos\gamma + \vec{k}\sin\gamma\right) \times \left[(\Omega\cos\gamma)\vec{j} + (\omega_o + \Omega\sin\gamma)\vec{k}\right]$$

$$\vec{\alpha} = \Omega\omega_o\cos\gamma\,\vec{i} = 68.4\vec{i}$$

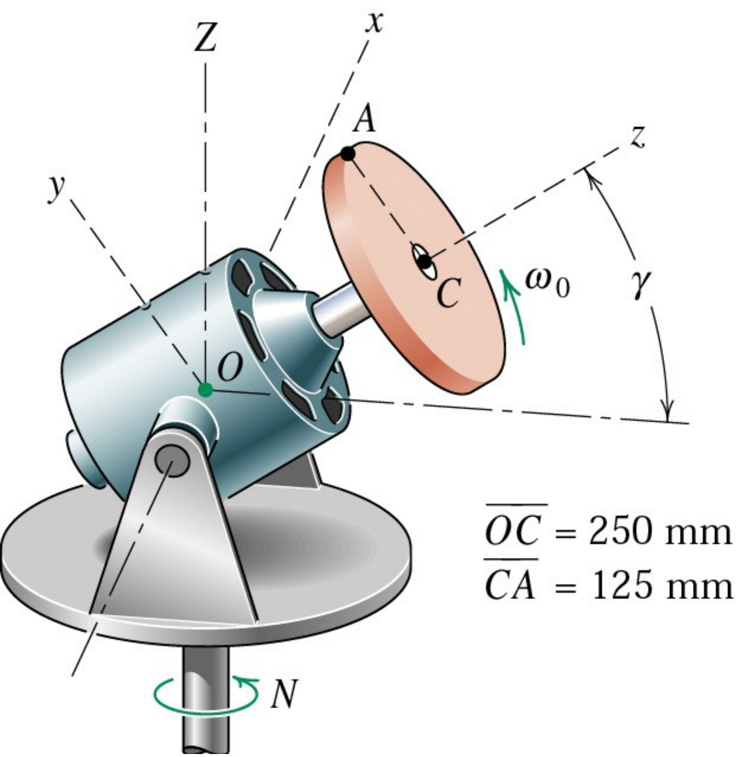
b) the space and body cone



$\overline{OC} = 250$ mm
$\overline{CA} = 125$ mm



(c) the velocity and acceleration of point A at the top of the disk



$$\vec{r}_{A/O} = 0.125\vec{j} + 0.25\vec{k}$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{v} = \pi\left(\sqrt{3}\vec{j} + 5\vec{k}\right) \times \left(0.125\vec{j} + 0.25\vec{k}\right)$$

$$\vec{v} = -0.192\pi\vec{i}$$

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

$$\vec{a} = -26.6\vec{j} + 11.83\vec{k}$$

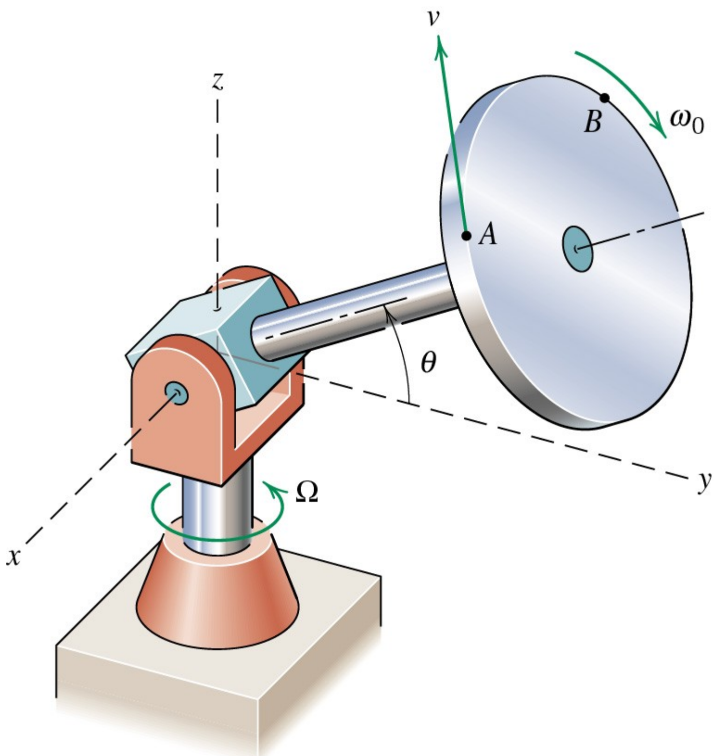
The rotor and shaft are mounted in a clevis which can rotate about the z-axis with an angular Ω. With Ω=0 and θ constant, the rotor has an angular velocity

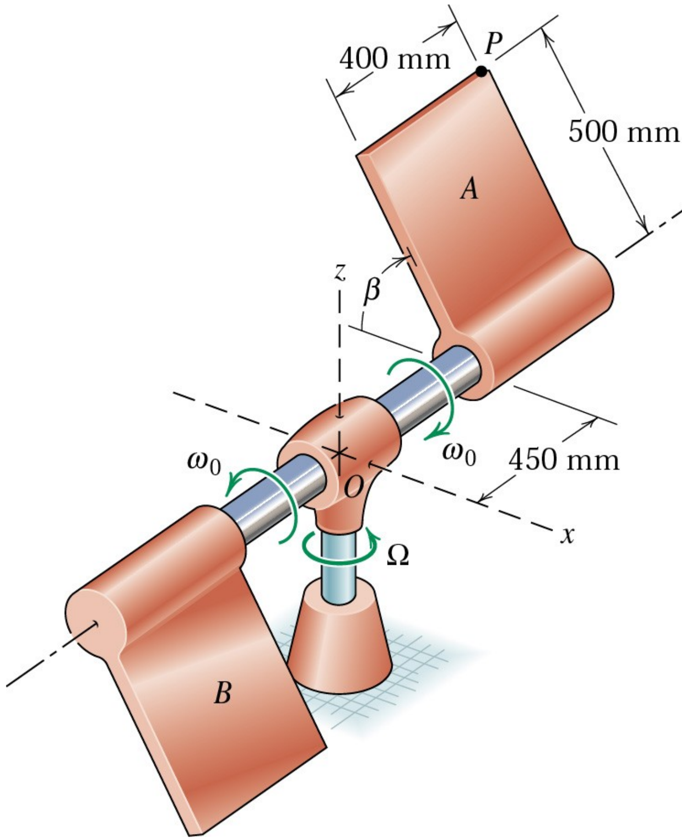
$$\vec{\omega}_o = -4\vec{j} - 3\vec{k} \quad rad/s.$$

Find the velocity v_A of point A on the rim if its position vector at the instant.

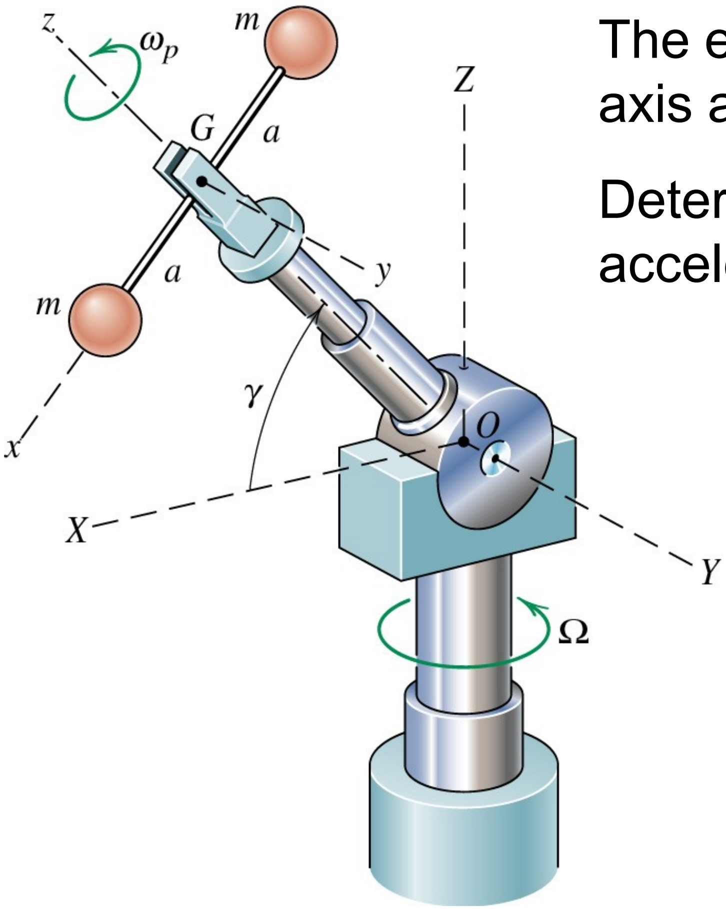
$$\vec{r} = 0.5\vec{i} + 1.2\vec{j} + 1.1\vec{k}$$

What is the rim speed v_B of any point B?





The panel assembly and attached *x*-*y*-*z* axes rotate with a constant angular velocity Ω=0.6 rad/s about the vertical *z*-axis. Simultaneously, the panels rotate about the *y*-axis as shown with a constant ω_o=2 rad/s. Determine the angular acceleration α of panel A and find the acceleration of point *P* for the instant when β=90°.



In manipulating the dumbbell, the jaws of robotic device have an angular velocity ω_P=2 rad/s about the axis OG with γ fixed at 60°.

The entire assembly rotates about the vertical Z-axis at the constant rate Ω=0.8 rad/s.

Determine the angular velocity and angular acceleration of the dumbbell.