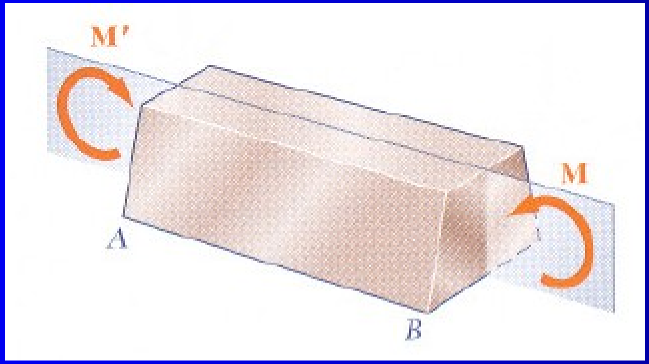
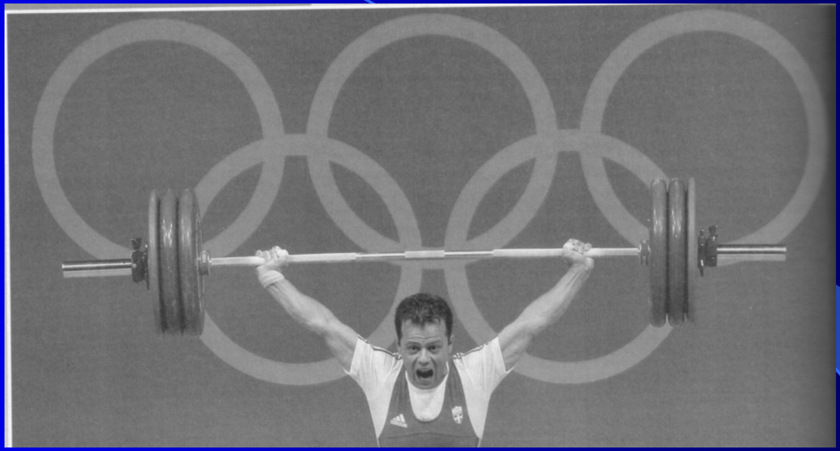
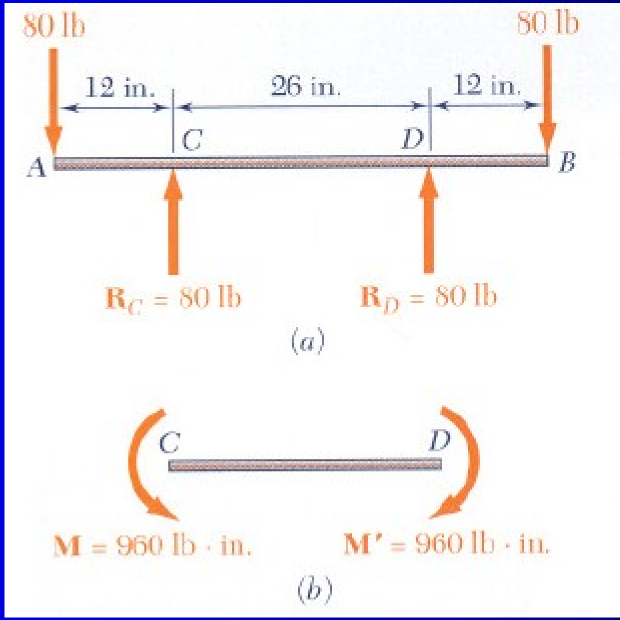
Pure Bending

- Axial
- Torsion
- Bending

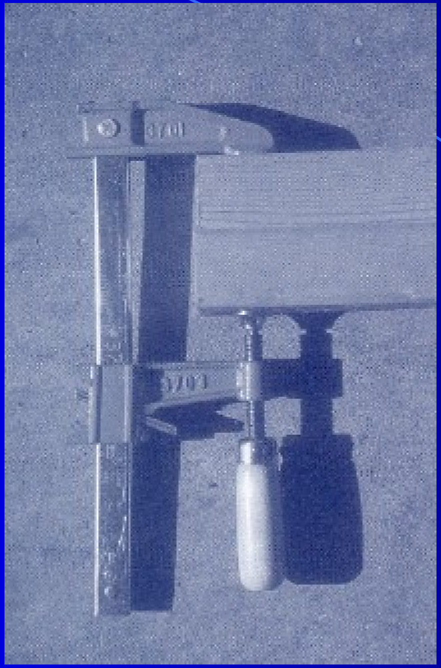


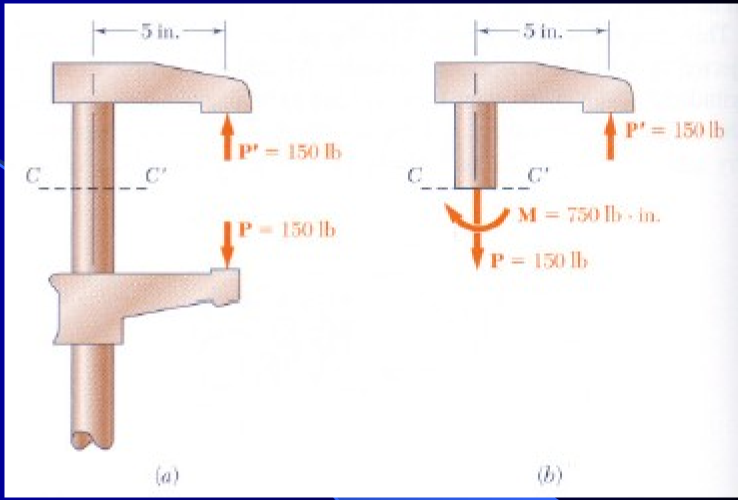
Introduction



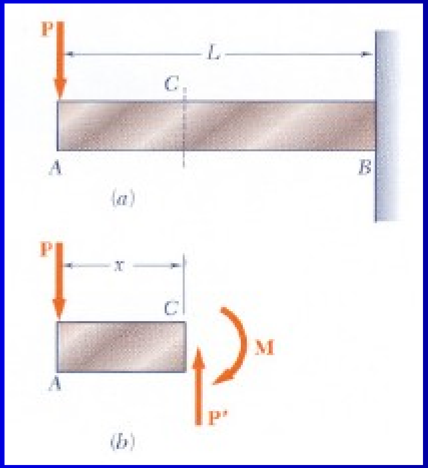


Eccentric Loading

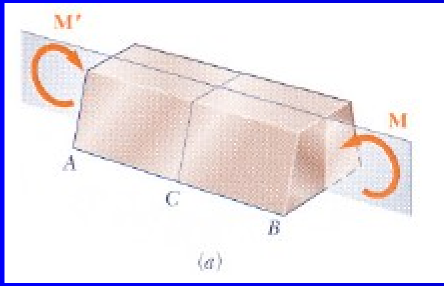


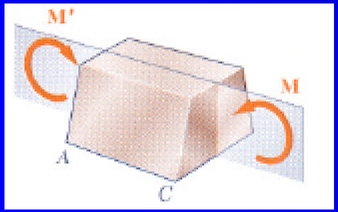


Pure Bending



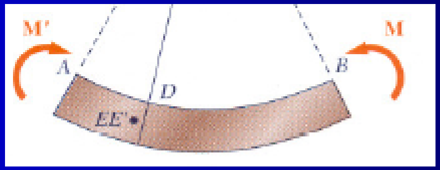
Symmetric Member in Pure Bending





M = Bending Moment

Sign Conventions for M:

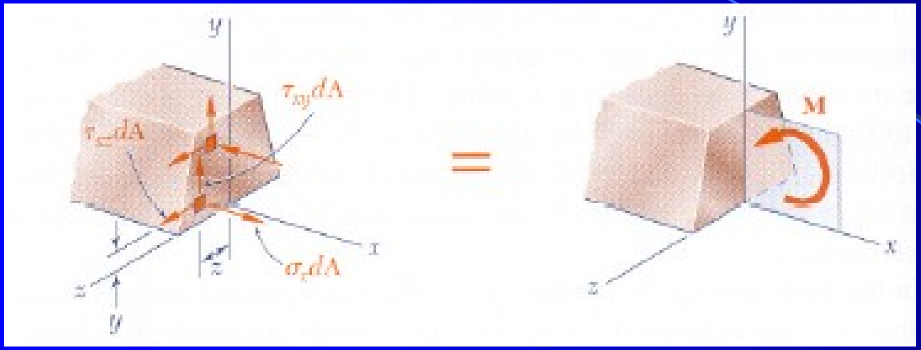


⊕ -- concave upward

⊖ -- concave downward

−

Force Analysis – Equations of Equilibrium



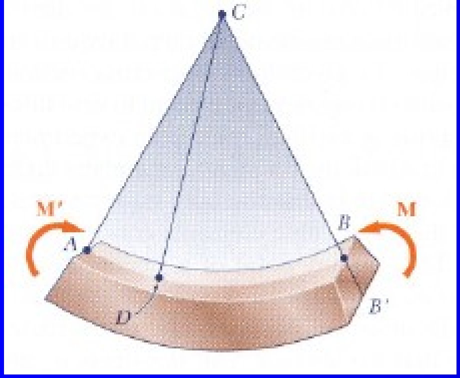
$$\tau_{xz} = \tau_{xy} = 0$$

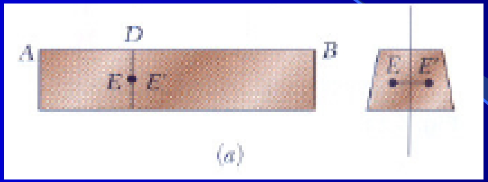
$\Sigma F_x = 0$ $$\int \sigma_x dA = 0 \quad (4.1)$$

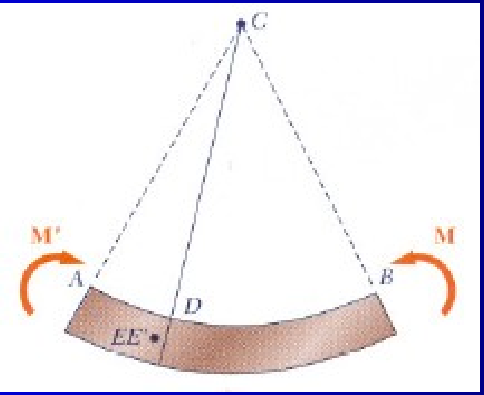
$\Sigma M_{y\text{-axis}} = 0$ $$\int z\sigma_x dA = 0 \quad (4.2)$$

$\Sigma M_{z\text{-axis}} = 0$ $$\int (-y\sigma_x dA) = M \quad (4.3)$$

Deformation in a Symmetric Member in Pure Bending







Plane CAB is the Plane of Symmetry

Assumptions of Beam Theory:

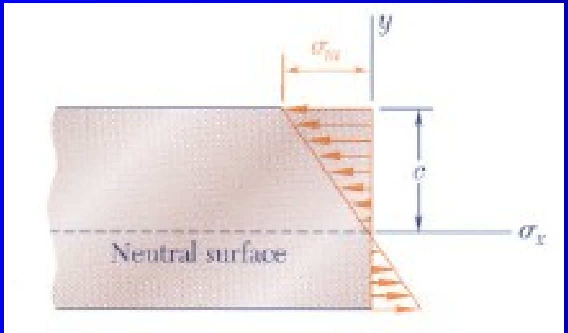
1. Any cross section ⊥ to the beam axis remains plane
2. The plane of the section passes through the center of curvature (Point C).

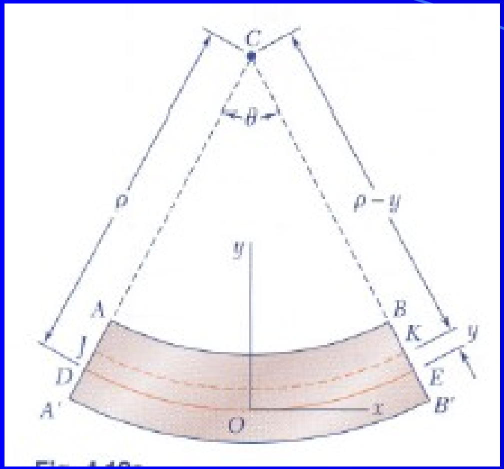
The Assumptions Result in the Following Facts:

1. $\tau_{xy} = \tau_{xz} = 0 \rightarrow \gamma_{xy} = \gamma_{xz} = 0$
2. $\sigma_y = \sigma_z = \tau_{yz} = 0$

The only non-zero stress: $\sigma_x \neq 0 \rightarrow$ **Uniaxial Stress**

The Neutral Axis (surface) : $\sigma_x = 0$ & $\varepsilon_x = 0$





$$L = \rho\theta$$ Line DE (4.4)

Where ρ = radius of curvature

θ = the central angle

$$L' = (\rho - y)\theta$$ Line JK (4.5)

Before deformation: DE = JK

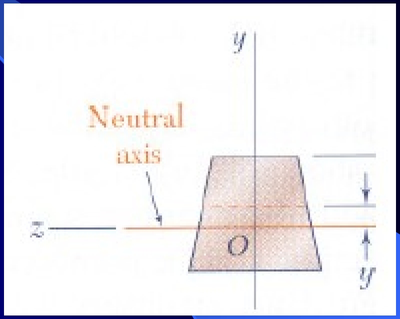
Therefore, $$\delta = L' - L$$ (4.6)

$$\delta = (\rho - y)\theta - \rho\theta = -y\theta$$

The Longitudinal Strain $\varepsilon_x = \frac{\Delta\ell}{\ell_o}$

$$\varepsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta}$$

$$\varepsilon_x = -\frac{y}{\rho} \qquad \textbf{(4.8)}$$



ε_x varies linearly with the distance y from the neutral surface

The max value of ε_x occurs at the top or the bottom fiber:

$$\varepsilon_m = \frac{c}{\rho} \qquad (4.9)$$

Combining Eqs (4.8) & (4.9) yields

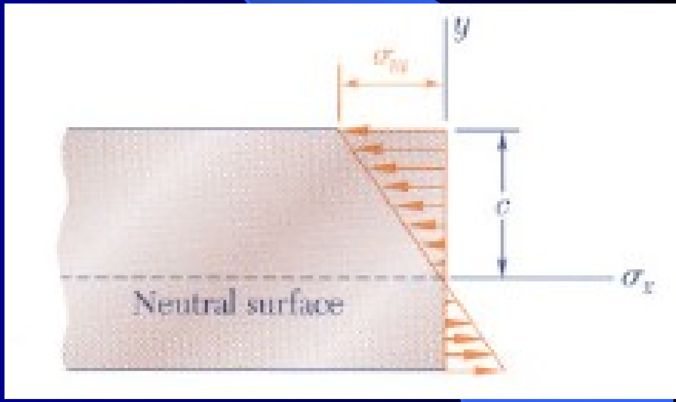
$$\varepsilon_x = -\frac{y}{c}\varepsilon_m \qquad (4.10)$$

Stresses and Deformation is in the Elastic Range

For elastic response – Hooke's Law

$$\sigma_x = E\varepsilon_x \qquad (4.11)$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m \qquad (4.10)$$



$$E\varepsilon_x = -\frac{y}{c}(E\varepsilon_m)$$

Therefore,

$$\sigma_x = -\frac{y}{c}\sigma_m = -\frac{y}{c}\sigma_{\max} \qquad (4.12)$$

Based on Eq. (4.1)

$$\int \sigma_x dA = 0 \qquad \textbf{(4.1)}$$

$$\sigma_x = -\frac{y}{c}\sigma_m = -\frac{y}{c}\sigma_{\max} \qquad \textbf{(4.12)}$$

$$\int \sigma_x dA = \int \left(-\frac{y}{c}\sigma_m\right) dA = -\frac{\sigma_m}{c}\int y dA = 0$$

Hence,

$$\int y dA = \text{first moment of area} = 0$$

(4.13)

Therefore,

Within elastic range, the neutral axis passes through the centroid of the section.

According to Eq. (4.3) $$\sigma_x = -\frac{y}{c}\sigma_m \qquad \textbf{(4.3)}$$

and $$\int(-y\sigma_x dA) = M \qquad \textbf{(4.12)}$$

It follows $$\int(-y)\left(-\frac{y}{c}\sigma_m\right)dA = M$$

or $$\frac{\sigma_m}{c}\int y^2 dA = M \qquad \textbf{(4.14)}$$

Since $$I = \int y^2 dA$$

Eq. (4.24) $$\frac{\sigma_m}{c} \int y^2 dA = M$$

can be written as

$$\sigma_m = \frac{Mc}{I}$$ **Elastic Flexure Formula (4.15)**

At any distance y from the neutral axis:

$$\sigma_x = -\frac{My}{I}$$ **Flexural Stress (4.16)**

If we define

$$\text{Elastic section modulus} = S = \frac{I}{c} \qquad \textbf{(4.17)}$$

Eq. (4.15) can be expressed as

$$\sigma_m = \frac{M}{S} \qquad \textbf{(4.18)}$$

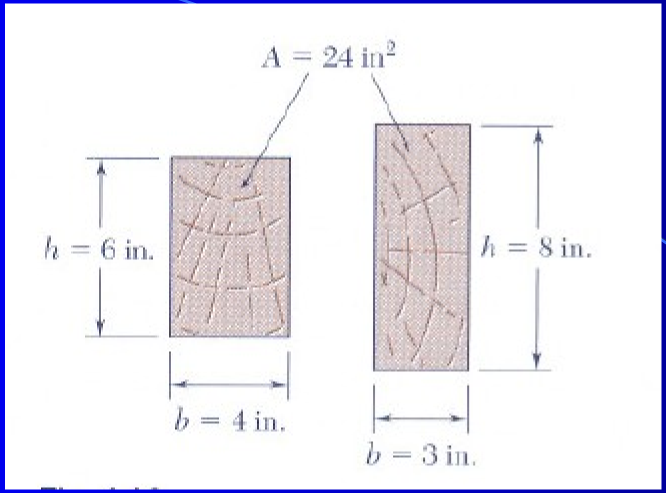
Solving Eq. (4.9) $$\varepsilon_m = \frac{c}{\rho} \qquad (4.9)$$

$$\rightarrow \quad \frac{1}{\rho} = \frac{\varepsilon_m}{c} \qquad \varepsilon_m = \frac{\sigma_m}{E}$$

$$\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I} \qquad [\because \quad \sigma_m = \frac{Mc}{I}]$$

Finally, we have

$$\frac{1}{\rho} = \frac{M}{EI} \qquad (4.21)$$



$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{\frac{h}{2}} = \frac{1}{6}bh^2 = \frac{1}{6}Ah$$

Deformation in a Transverse Cross Section

<u>**Assumption in Pure Bending of a Beam:**</u>

The transverse cross section of a beam remains "plane".

However, this plane may undergo *in-plane* deformations.

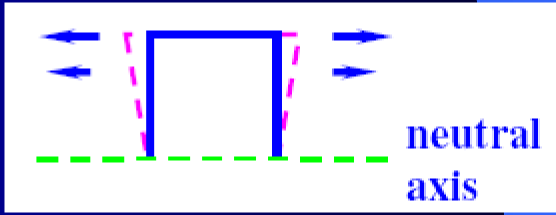
A. Material above the neutral surface (y>0), $\sigma_x = \ominus, \varepsilon_x = \ominus$

$$\varepsilon_y = -\nu\varepsilon_x \qquad \varepsilon_z = -\nu\varepsilon_x$$

Since $$\varepsilon_x = -\frac{y}{\rho} \qquad \text{(4.8)}$$

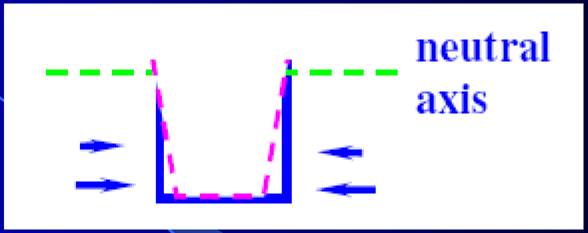
Hence, $$\varepsilon_y = \frac{\nu y}{\rho} \qquad \varepsilon_z = \frac{\nu y}{\rho} \qquad \text{(4.22)}$$

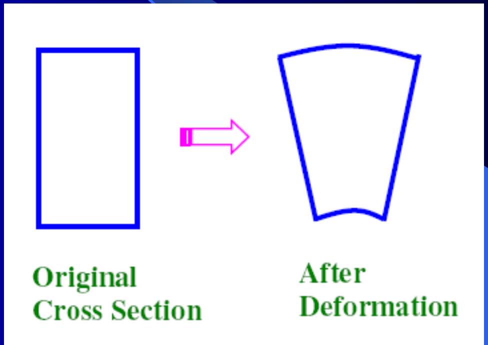
Therefore, $\varepsilon_y = \oplus, \varepsilon_z = \oplus$



Material below the neutral surface (y<0), $\sigma_x = \oplus, \varepsilon_x = \oplus$

$$\Longrightarrow \quad \varepsilon_y = \ominus \quad \varepsilon_z = \ominus$$





As a consequence,

Analogous to Eq. (4.8)

$$\varepsilon_x = -\frac{y}{\rho} \quad \rightarrow \quad \rho = -\frac{y}{\varepsilon_x}$$

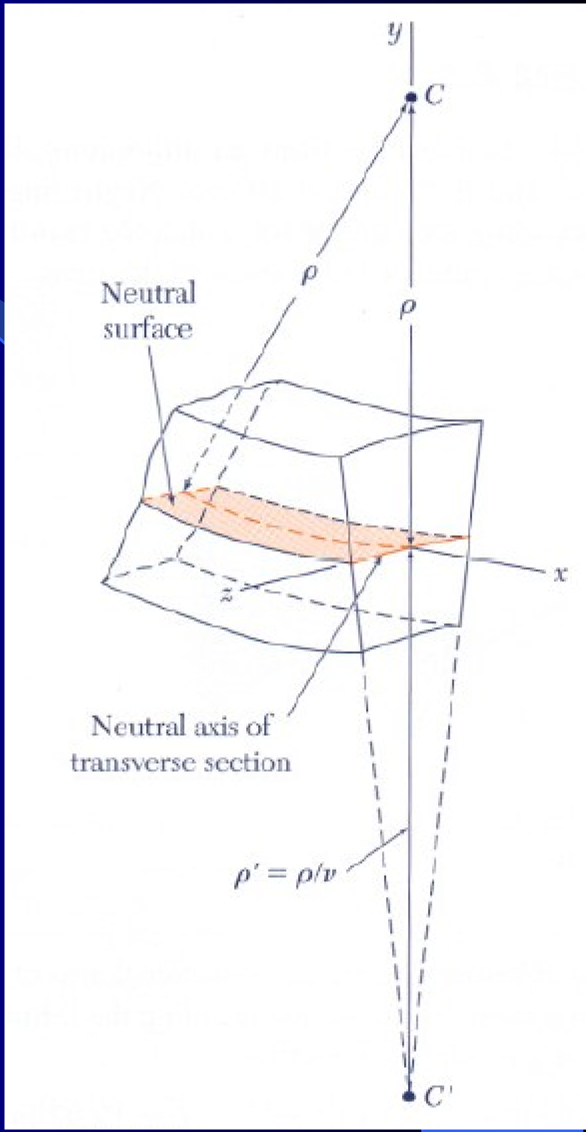
For the transverse plane:

$$\rho' = -\frac{y}{\varepsilon_y} = \frac{y}{\upsilon \varepsilon_x} = \frac{1}{\upsilon}\frac{y}{\varepsilon_x} = \frac{\rho}{\upsilon}$$

ρ = radius of curvature,

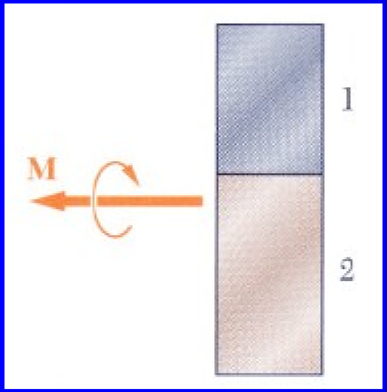
1/ρ = curvature

$$\textbf{Anticlastic curvature} = \frac{1}{\rho'} = \frac{\nu}{\rho} \qquad (4.23)$$



Bending of Members Made of Several Materials

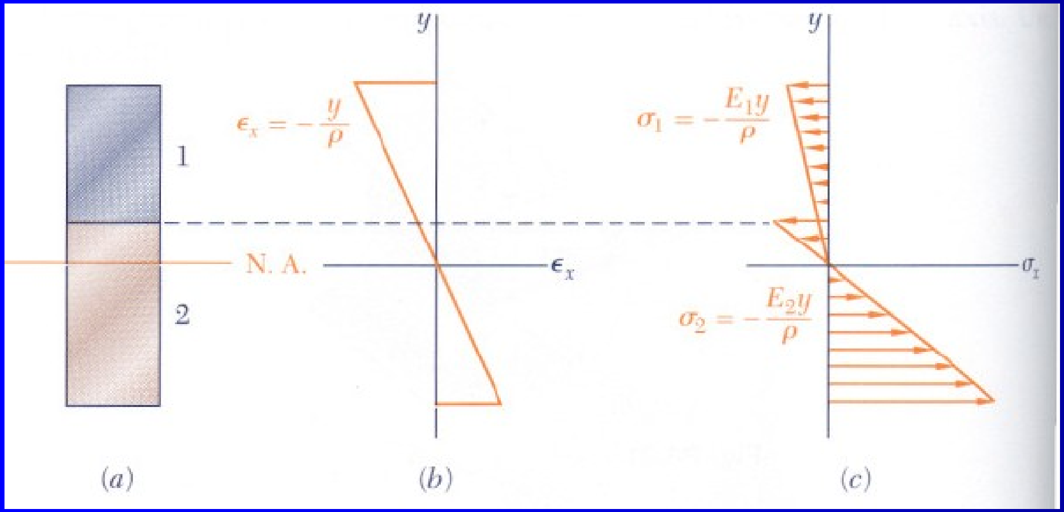
(Composite Beams)



From Eq. (4.8) $$\varepsilon_x = -\frac{y}{\rho}$$

For Material 1: $$\sigma_1 = E_1\varepsilon_x = -\frac{E_1 y}{\rho}$$

For Material 2: $$\sigma_2 = E_2\varepsilon_x = -\frac{E_2 y}{\rho}$$

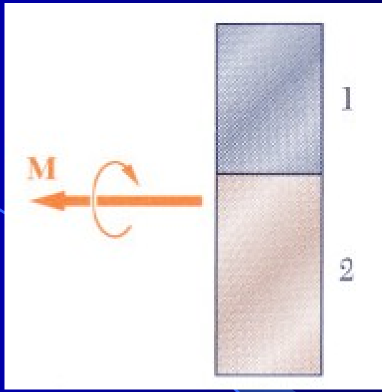


$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA$$

$$dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

Designating $E_2 = nE_1$

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n dA)$$



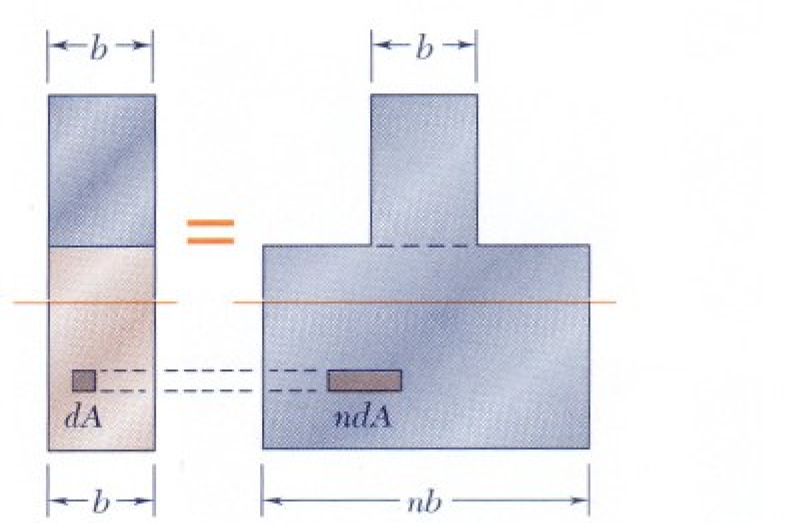


Fig. 4.25 Transformed section for composite bar.

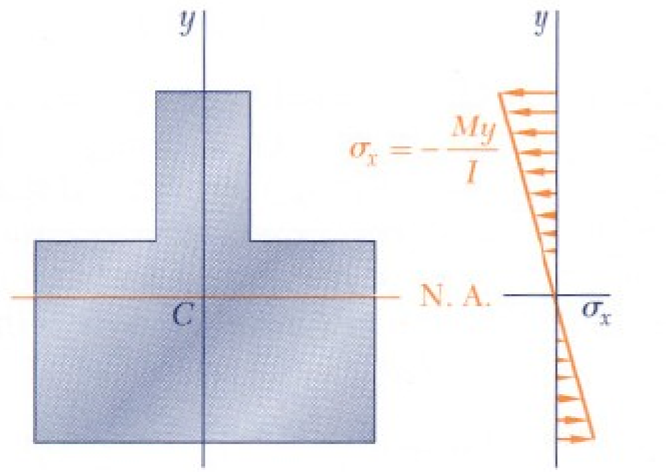


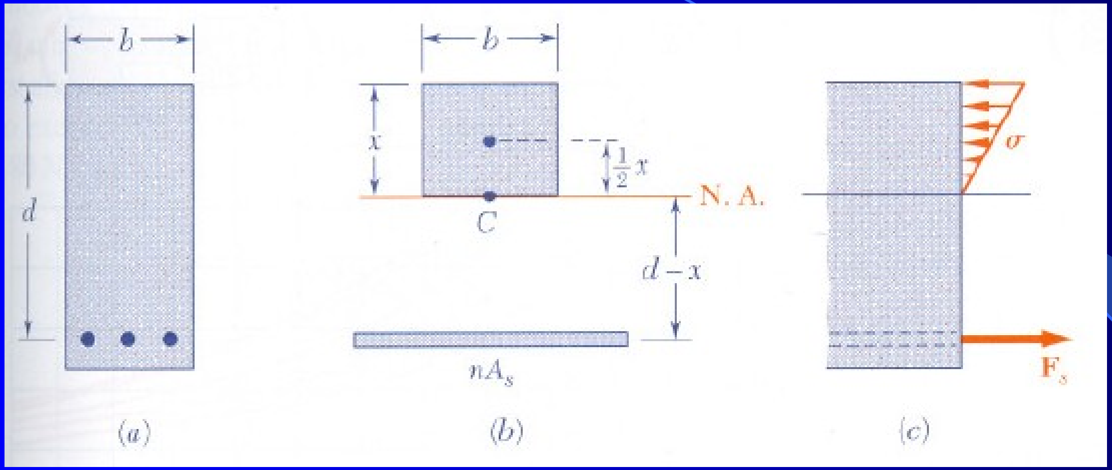
Fig. 4.26 Distribution of stresses in transformed section.

$$\sigma_x = -\frac{My}{I}$$

Notes:

1. The neutral axis is calculated based on the transformed section.
2. $\sigma_2 = n\sigma_x$
3. I = the moment of inertia of the transformed section
4. Deformation -- $\frac{1}{\rho} = \frac{M}{E_1 I}$

Beam with Reinforced Members:

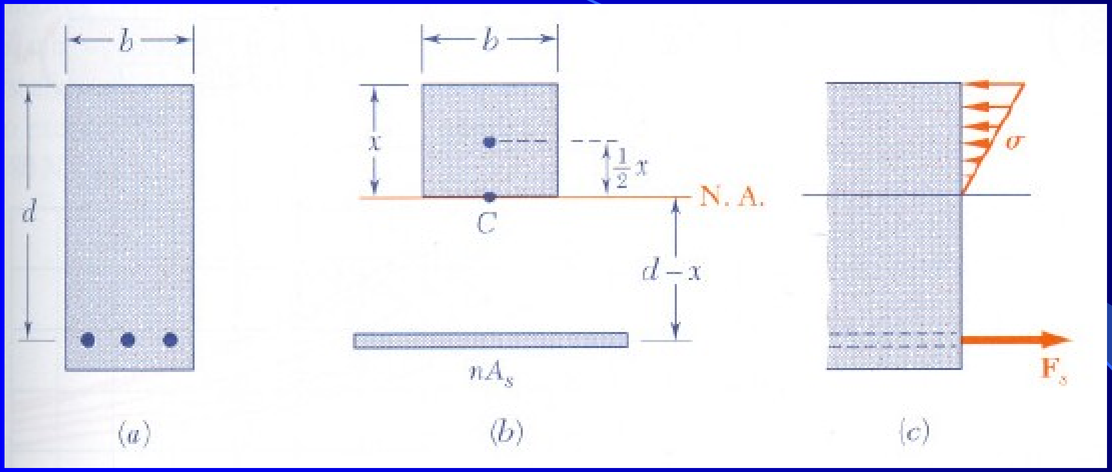


A_s = area of steel, A_c = area of concrete

E_s = modulus of steel, E_c = modulus of concrete

$n = E_s/E_c$

Beam with Reinforced Members:



A_s = area of steel, A_c = area of concrete

E_s = modulus of steel, E_c = modulus of concrete

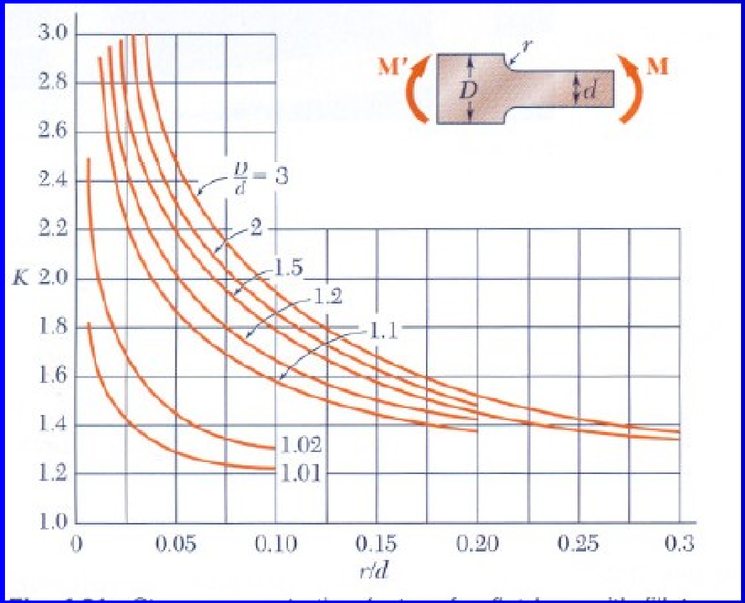
$n = E_s/E_c$

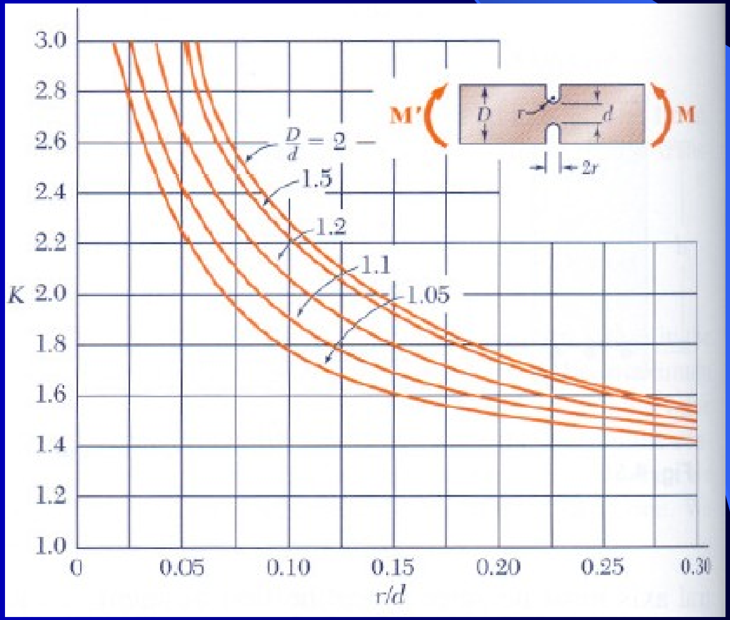
$$(bx)\frac{x}{2} - nA_s(d - x) = 0$$

$$\frac{1}{2}bx^2 + nA_s x - nA_s d = 0$$ → determine the N.A.

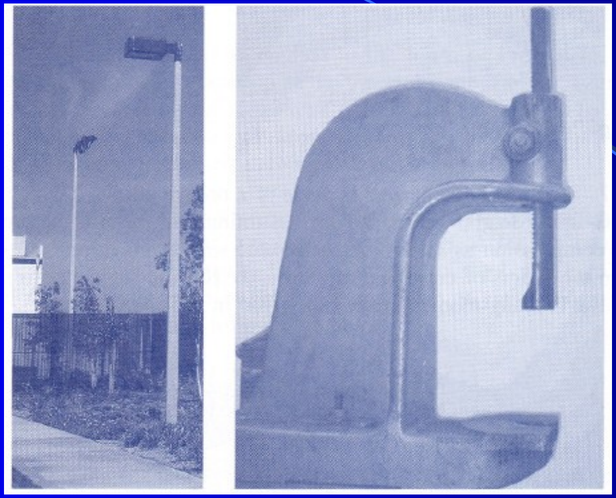
Stress Concentrations

$$\sigma_m = K\frac{Mc}{I}$$





Eccentric Axial Loading in a Plane of Symmetry



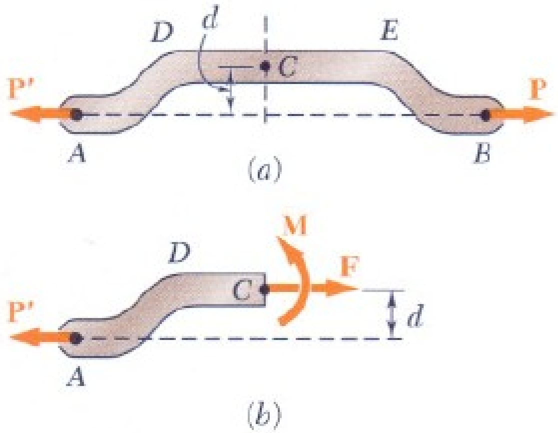
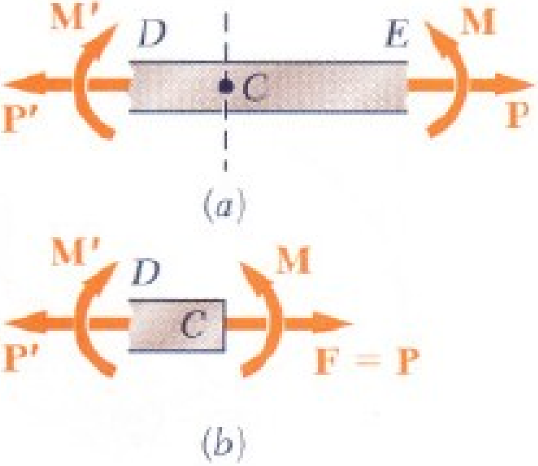
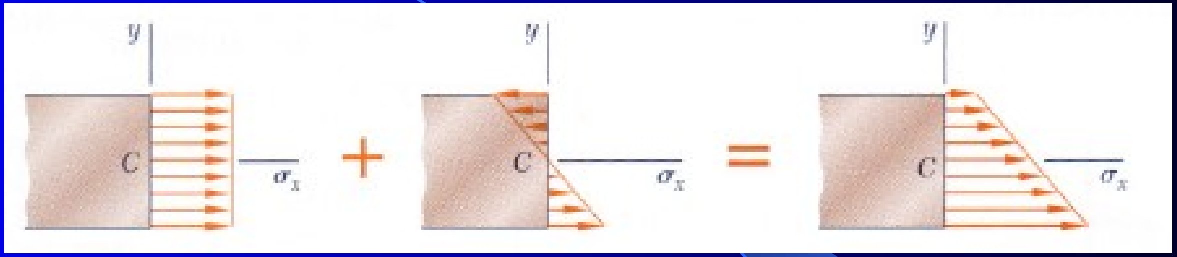


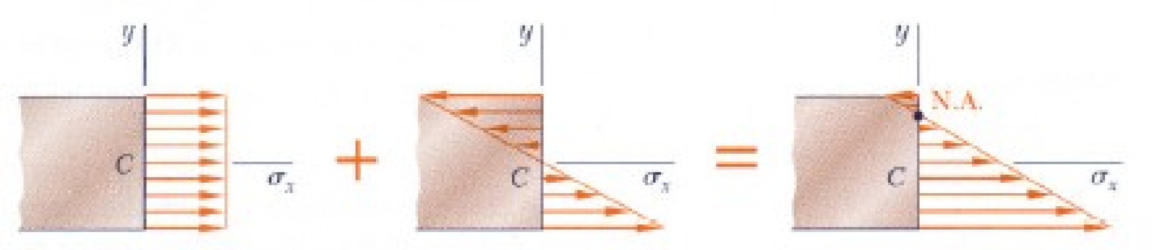
Fig. 4.48



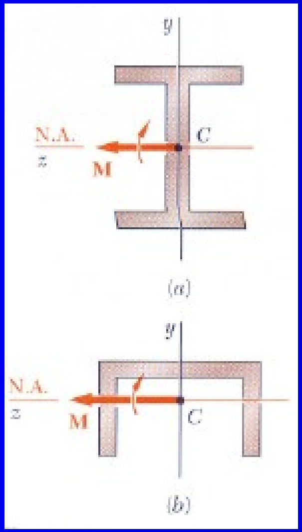
$$\sigma_x = (\sigma_x)_{centric} + (\sigma_x)_{bending}$$



$$\sigma_x = \frac{P}{A} - \frac{My}{I}$$



Unsymmetric Bending

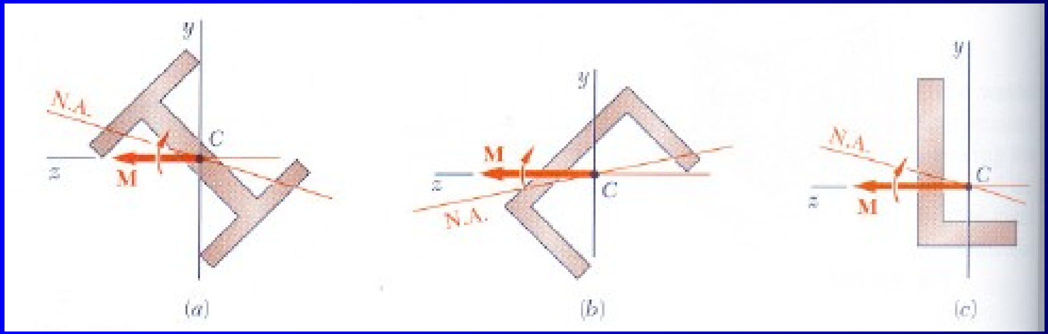


-- Two planes of symmetry

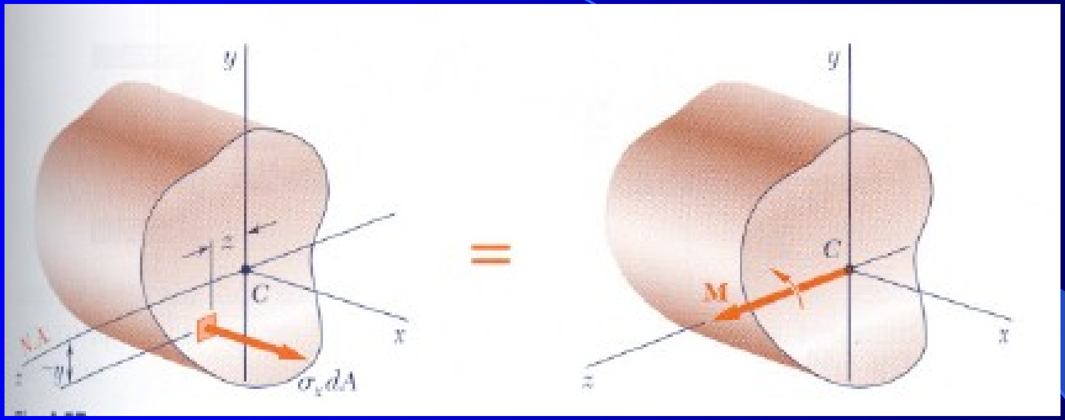
y – axis & z-axis

-- Single plane of symmetry – y-axis

--M coincides with the N.A.



For an arbitrary geometry + M applies along the N.A

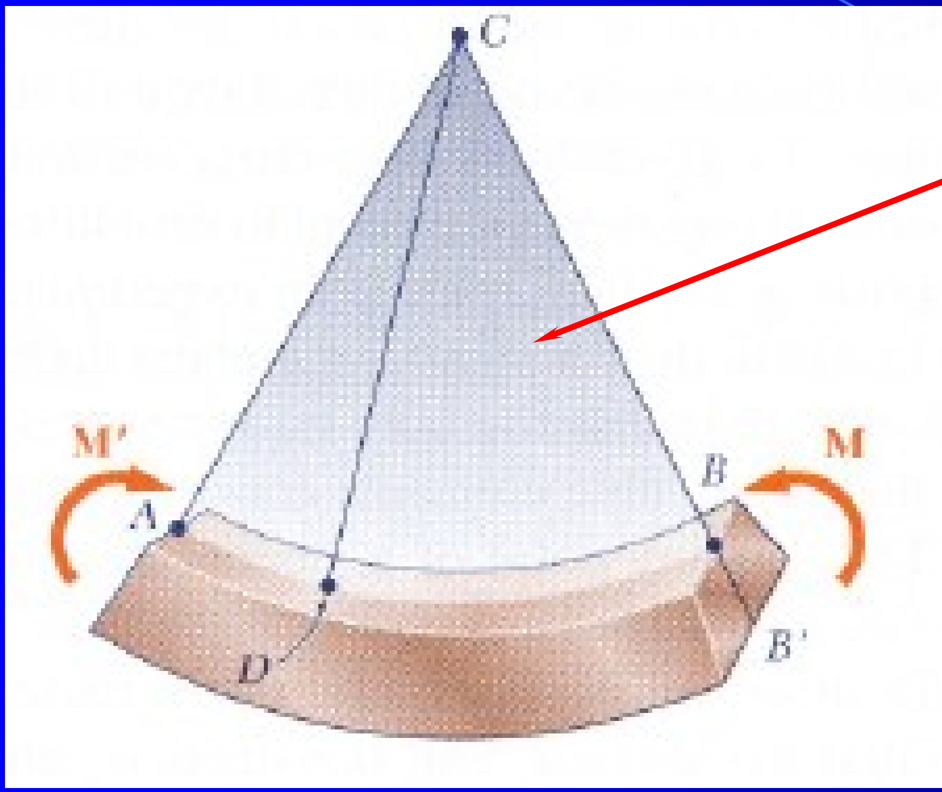


$\Sigma F_x = 0$ $\int \sigma_x dA = 0$ **(the Centroid = the N.A.)** **(4.1)**

$\Sigma M_y = 0$ $\int z\sigma_x dA = 0$ (moment equilibrium) **(4.2)**

$\Sigma M_z = 0$ $\int -(y\sigma_x dA) = M$ (moment equilibrium) **(4.3)**

Substituting $\sigma_x = -\frac{\sigma_m y}{c}$ **into Eq. (4.2)**


C
M'
M
A
B
D
B'
Plane of symmetry

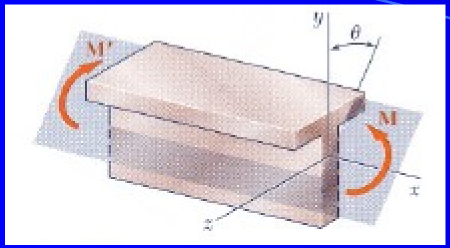
We have $$\int z\left(-\frac{\sigma_m y}{c}\right)dA = 0 \quad or \quad -\frac{\sigma_m}{c}\int z(y)dA = 0$$

or $$\int yz dA = I_{yz} = 0$$ (knowing σ_m/c = constant)

I_{yz} = 0 indicates that y- and z-axes are the principal centroid of the cross section.

Hence, the N.A. coincides with the M-axis.

If the axis of M coincides with the principal centroid axis, the superposition method can be used.



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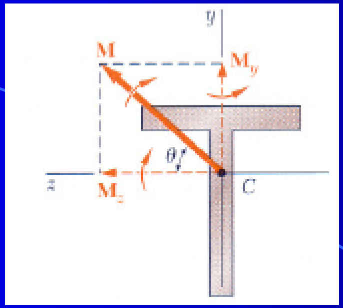

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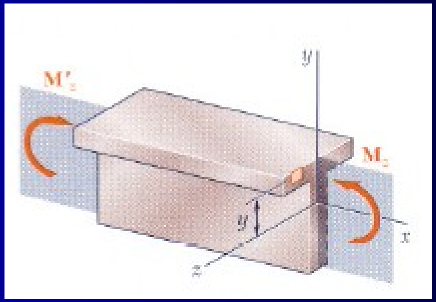
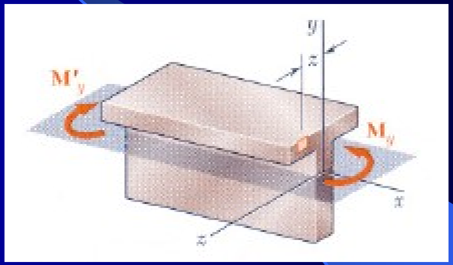


Case A

+



Case B

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

For Case A $$\sigma_x = -\frac{M_z y}{I_z} \qquad (4.53)$$

For Case B $$\sigma_x = +\frac{M_y z}{I_y} \qquad (4.54)$$

For the combined cases : $$\boldsymbol{\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}} \qquad (4.55)$$

The N.A. is the surface where $\sigma_x = 0$. By setting $\sigma x = 0$ in Eq. (4.55), one has

$$-\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

Solving for y and substituting for M_z and M_y from Eq. (4.52),

$$y = \left(\frac{I_z}{I_y}\tan\theta\right) z \qquad \textbf{(4.56)}$$

This is equivalent to $y/z = m = slope = \left(\frac{I_z}{I_y}\right)\tan\theta$

The N.A. is an angle ϕ from the z-axis:

$$\tan\phi = \frac{I_z}{I_y}\tan\theta \qquad \textbf{(4.57)}$$

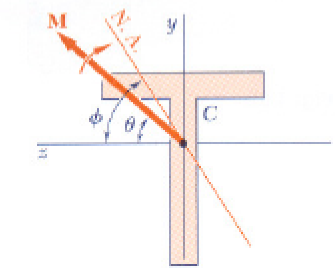
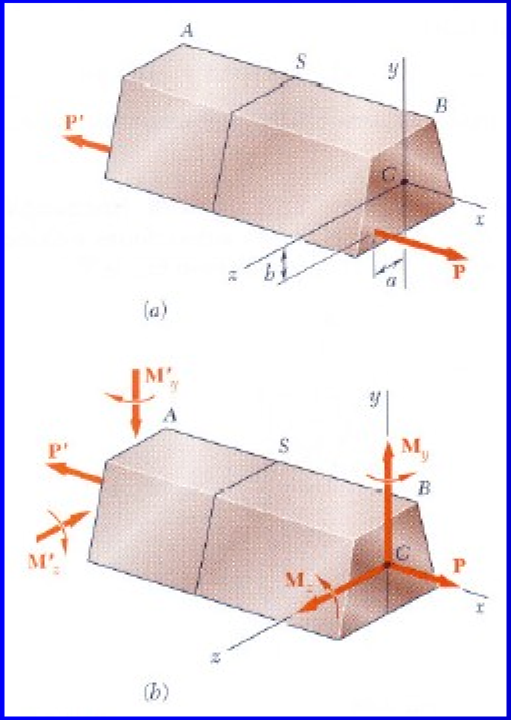


Fig. 4.65

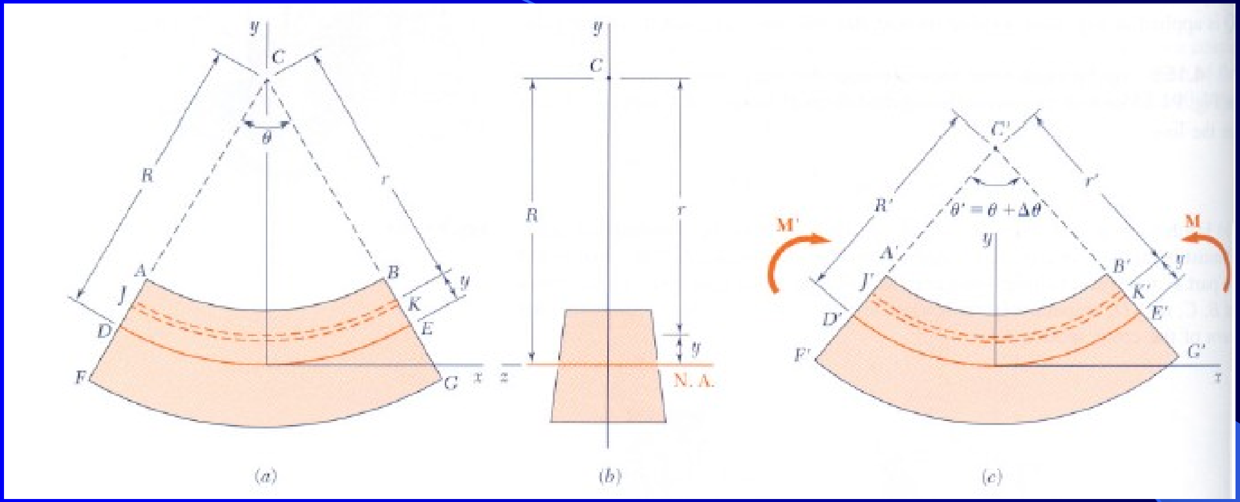
General Case of Eccentric Axial loading



$$\sigma_x = \frac{P}{A} - \frac{M_Z y}{I_z} + \frac{M_y z}{I_y} \qquad (4.58)$$

$$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A} \qquad (4.58)$$

Bending of Curved Members



Before bending **After bending**

Length of N.A. before and after bending $$R\theta = R'\theta' \quad (4.59)$$

The elongation of JK line $$\delta = r'\theta' - r\theta \quad (4.60)$$

Since $$r = R - y \qquad r' = R' - y \quad (4.61)$$

We have $$\delta = (R' - y)\theta' - (R - y)\theta$$

If we define $\theta' - \theta = \Delta\theta$ and knowing $R\,\theta = R'\,\theta'$, thus

$$\delta = -y\Delta\theta \qquad (4.62)$$

Based on the definition of strain, we have

$$\varepsilon_x = \frac{\delta}{r\theta} = -\frac{y\Delta\theta}{r\theta} \qquad (4.63)$$

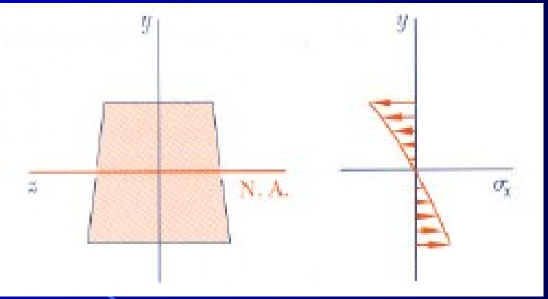
Substituting $r = R - y$ into the above equation,

$$\varepsilon_x = -\frac{\Delta\theta}{\theta}\frac{y}{R-y} \qquad (4.64)$$

Also, $\boldsymbol{\sigma_x = E\,\varepsilon_x}$

$$\sigma_x = -\frac{E\Delta\theta}{\theta}\frac{y}{R-y} \qquad (4.65)$$

Plotting $\sigma_x = -\frac{E\Delta\theta}{\theta}\frac{y}{R-y}$ $\rightarrow$



$\Rightarrow$ **σ_x is not a linear function of y.**

Since $r = R - y$ $\rightarrow$ y = R – r, therefore,

$$\sigma_x = -\frac{E\Delta\theta}{\theta}\frac{R-r}{r}$$

Substituting this eq. into Eq. (4.1) $\int \sigma_x dA = 0$

$$-\int \frac{E\Delta\theta}{\theta}\frac{R-r}{r}dA = 0 \quad \text{and} \quad -\frac{E\Delta\theta}{\theta}\int \frac{R-r}{r}dA = 0$$

$\Rightarrow$ $$\int \frac{R-r}{r}dA = 0 \quad \text{or} \quad R\int \frac{dA}{r} - \int dA = 0 \quad \left(\frac{E\Delta\theta}{\theta} = \text{cos}\,t.\right)$$

Therefore, R can be determined by the following equation:

$$R = \frac{A}{\int \frac{dA}{r}} \qquad (4.66)$$

Or in an alternative format: $\frac{1}{R} = \frac{1}{A}\int \frac{1}{r} dA$

The centroid of the section is determined by

$$\bar{r} = \frac{1}{A}\int r dA \qquad (4.67)$$

(4.59)

Comparing Eqs. (4.66) and (4.67), we conclude that:

The N.A. axis does not pass through the Centroid of the cross section.

$\Sigma M_z = M \Rightarrow$ $$\int \frac{E\Delta\theta}{\theta}\frac{R-r}{r} y\,dA = M$$

Since $y = R - r$, it follows

$$\frac{E\Delta\theta}{\theta}\int \frac{(R-r)^2}{r} dA = M$$

or

$$\frac{E\Delta\theta}{\theta}[R^2 \int \frac{dA}{r} - 2RA + \int r\,dA] = M$$

Recalling Eqs. (4-66) and (4.67), we have

$$\frac{E\Delta\theta}{\theta}(RA - 2RA + \bar{r}A) = M$$

Finally,

$$\frac{E\Delta\theta}{\theta} = \frac{M}{A(\bar{r} - R)} \qquad (4.68)$$

By defining $e = \bar{r} - R$, the above equation takes the new form

$$\frac{E\Delta\theta}{\theta} = \frac{M}{Ae} \qquad (4.69)$$

Substituting this expression into Eqs. (4.64) and (4-65), we have

$$\sigma_x = -\frac{My}{Ae(R-y)} \quad \text{and} \quad \sigma_x = \frac{M(r-R)}{Aer} \qquad (4.70, 71)$$

Determination of the change in curvature:

From Eq. (4.59) $\frac{1}{R'} = \frac{1}{R}\frac{\theta'}{\theta}$

Since $\theta' = \theta + \Delta\theta$ and from Eq. (4.69), one has

$$\frac{1}{R'} = \frac{1}{R}(1 + \frac{\Delta\theta}{\theta}) = \frac{1}{R}(1 + \frac{M}{EAe})$$

Hence, the change of curvature is

$$\frac{1}{R'} - \frac{1}{R} = \frac{M}{EAeR}$$

$$\sigma_x = -\frac{E\Delta\theta}{\theta}\frac{R-r}{r}$$

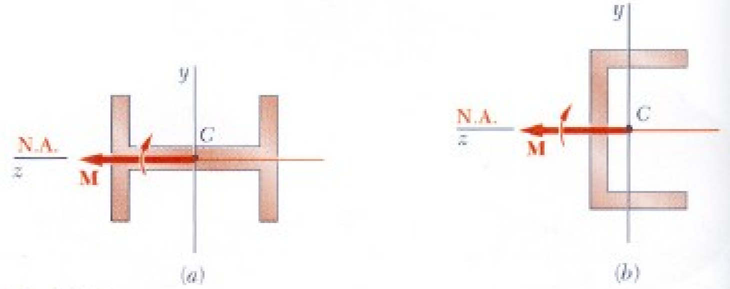


Fig. 4.58

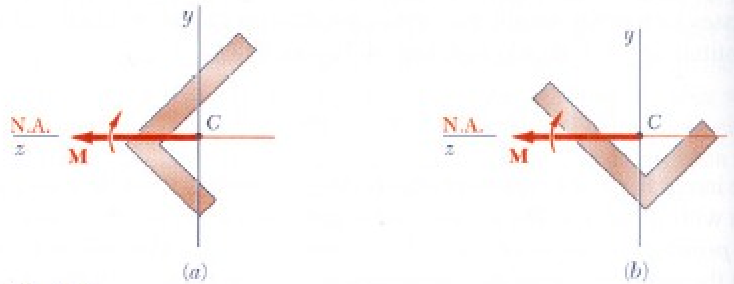


Fig. 4.59

$$\sigma_x = \frac{P}{A} - \frac{M_Z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_x = (\sigma_x)_{centric} + (\sigma_x)_{bending}$$

$$\sigma_x = \frac{P}{A} - \frac{My}{I}$$ **(4.58)**

(4.58)

$$\int z\left(-\frac{\sigma_m y}{c}\right)dA = 0$$ **(4.58)**

$$\int yz\,dA = 0$$ **(4.58)**

$$(bx)\frac{x}{2} - nA_s(d - x) = 0$$

$$\frac{1}{2}bx^2 + nA_s x - nA_s d = 0$$

$$\sigma_m = K\frac{Mc}{I}$$

$$M = -b\int_{-c}^{c} y\sigma_x dy$$

$$M = -2b\int_{0}^{c} y\sigma_x dy$$

$$R_B = \frac{M_U c}{I}$$

$$R = \frac{A}{\int_{r_1}^{r_2} \frac{dA}{r}} = \frac{bh}{\int_{r_1}^{r_2} \frac{bdr}{r}} = \frac{h}{\int_{r_1}^{r_2} \frac{dr}{r}}$$

$$R = \frac{h}{\ln \frac{r_2}{r_1}}$$

$$Z = \frac{M_p}{\sigma_Y} = \frac{bc^2\sigma_Y}{\sigma_Y} = bc^2 = \frac{1}{4}bh^2$$

$$S = \frac{1}{6}bh^2$$

$$k = \frac{Z}{S} = \frac{\frac{1}{4}bh^2}{\frac{1}{6}bh^2} = \frac{3}{2}$$

$$F = P$$

$$M = Pd$$

$$R_Y = \frac{1}{2} bc\sigma_Y$$

$$R_p = bc\sigma_Y$$

$$M_Y = (\frac{4}{3}c)R_Y = \frac{2}{3} bc^2\sigma_Y$$

$$M_p = cR_p = bc^2\sigma_Y$$

$$M_p = kM_Y$$

$$M_p = Z\sigma_Y$$

$$k = \frac{M_p}{M_Y} = \frac{Z\sigma_Y}{S\sigma_Y} = \frac{Z}{S}$$

$$M = bc^2\sigma_Y\left(1 - \frac{1}{3}\frac{y_Y^2}{c^2}\right)$$

$$M = \frac{3}{2}M_Y\left(1 - \frac{1}{3}\frac{y_Y^2}{c^2}\right)$$

$$M_p = \frac{3}{2}M_Y$$

$$y_Y = \varepsilon_Y \rho$$

$$c = \varepsilon_Y \rho_Y$$

$$\frac{y_Y}{c} = \frac{\rho}{\rho_Y}$$

$$M = \frac{3}{2}M_Y\left(1 - \frac{1}{3}\frac{\rho^2}{\rho_Y^2}\right)$$

$$M_Y = \frac{1}{c}\sigma_Y$$

$$\frac{I}{c} = \frac{b(2c)^3}{12c} = \frac{2}{3}bc^2$$

$$M_Y = \frac{2}{3}bc^2\sigma_Y$$

$$\sigma_x = -\frac{\sigma_Y}{y_Y}y$$

$$M = -2b\int_0^{y_Y} y\left(-\frac{\sigma_Y}{y_Y}y\right)dy - 2b\int_{y_Y}^{c} y(-\sigma_Y)dy$$

$$= \frac{2}{3}by_Y^2\sigma_Y + bc^2\sigma_Y - by_Y^2\sigma_Y$$

$$\tan\phi = \frac{I_z}{I_y}\tan\theta$$

(4.58)

$$\frac{M_z}{I_z}y - \frac{M_y}{I_y}z = \frac{P}{A}$$

(4.58)

(4.58)