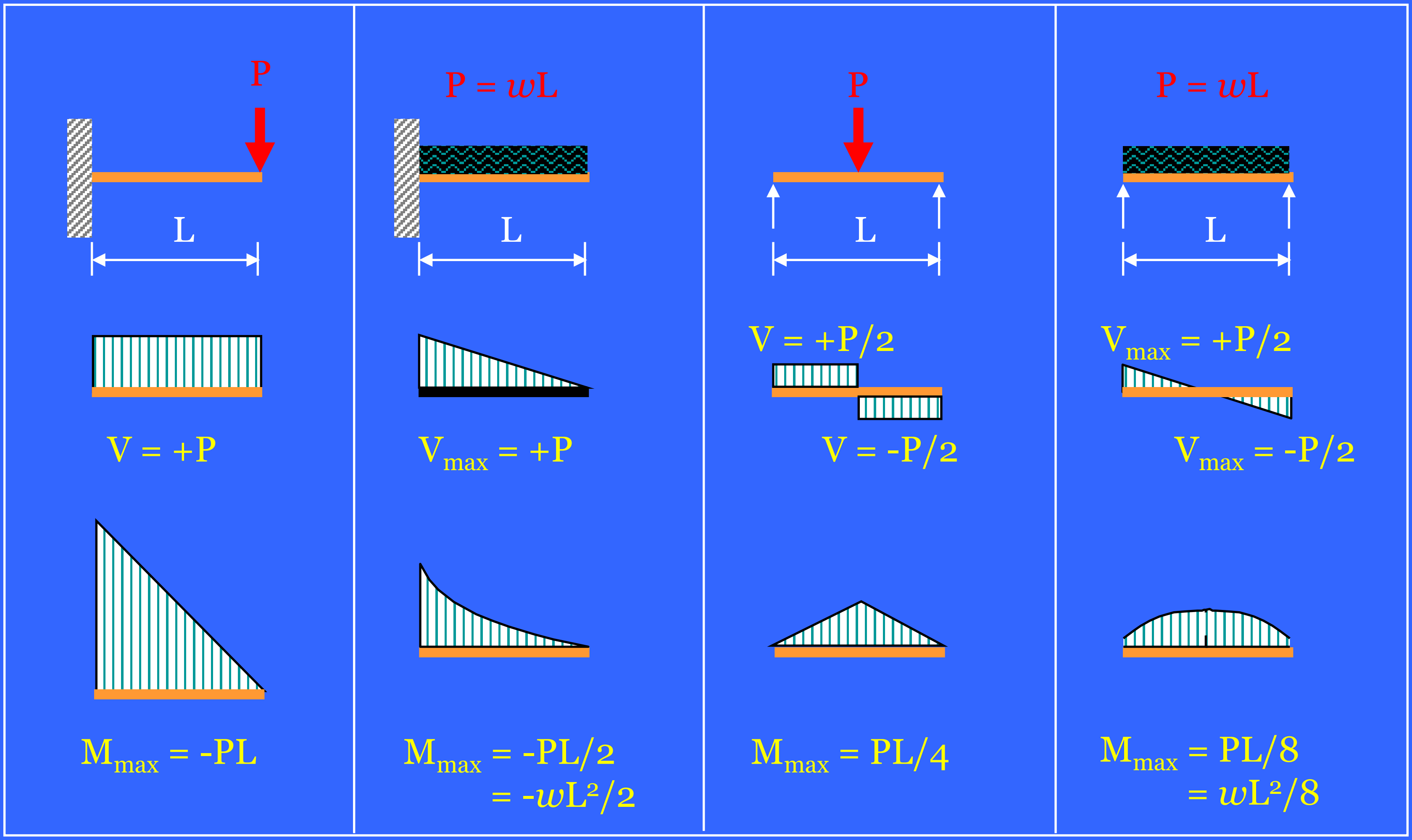
Shear Force and Bending Moment

- ✓ **Shear Force:** is the algebraic sum of the vertical forces acting to the left or right of a cut section along the span of the beam

- ✓ **Bending Moment:** is the algebraic sum of the moment of the forces to the left or to the right of the section taken about the section

SFD & BMD Simply Supported Beams



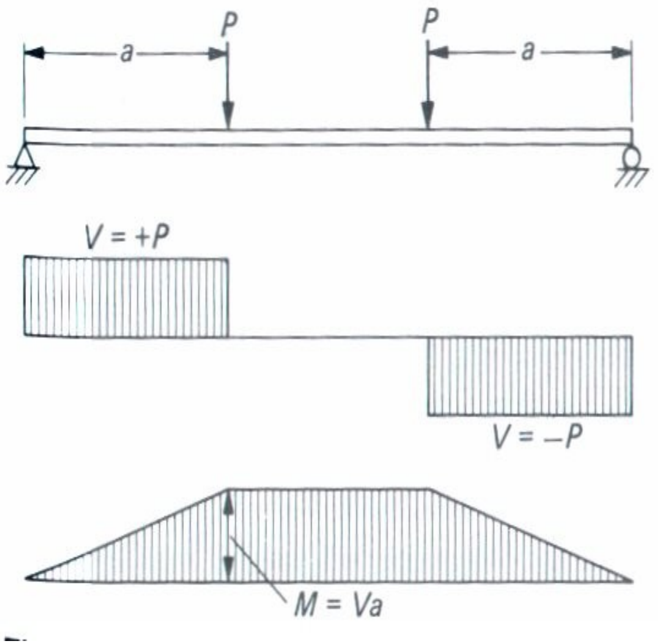
Longitudinal strain
Longitudinal stress
Location of neutral surface
Moment-curvature equation

Bending of Beams

- ✓ It is important to distinguish between pure bending and non-uniform bending.

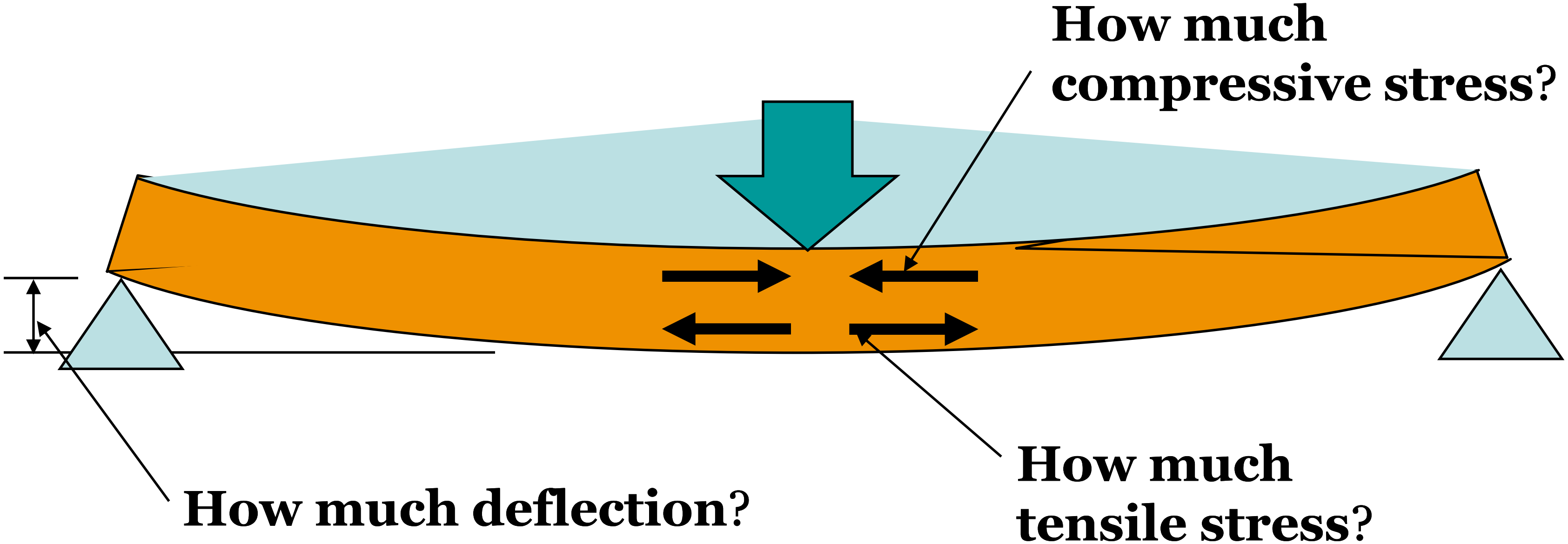
- ✓ Pure bending is the deformation of the beam under a constant bending moment. Therefore, pure bending occurs only in regions of a beam where the shear force is zero, because V = dM/dx.

- ✓ Non-uniform bending is deformation in the presence of shear forces, and bending moment changes along the axis of the beam.



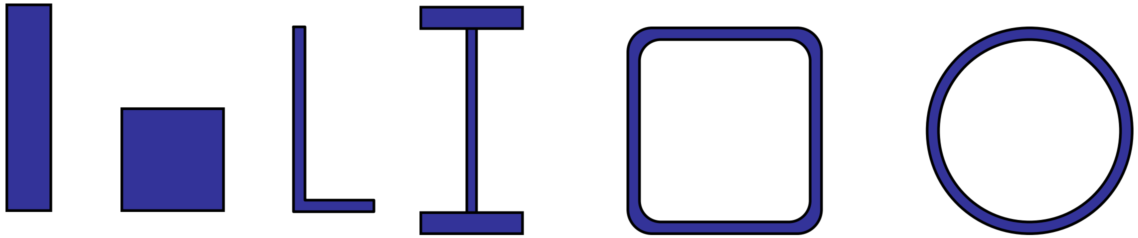
What the Bending Moment does to the Beam

- ✓ **Causes compression on one face and tension on the other**
- ✓ **Causes the beam to deflect**



How to Calculate the Bending Stress

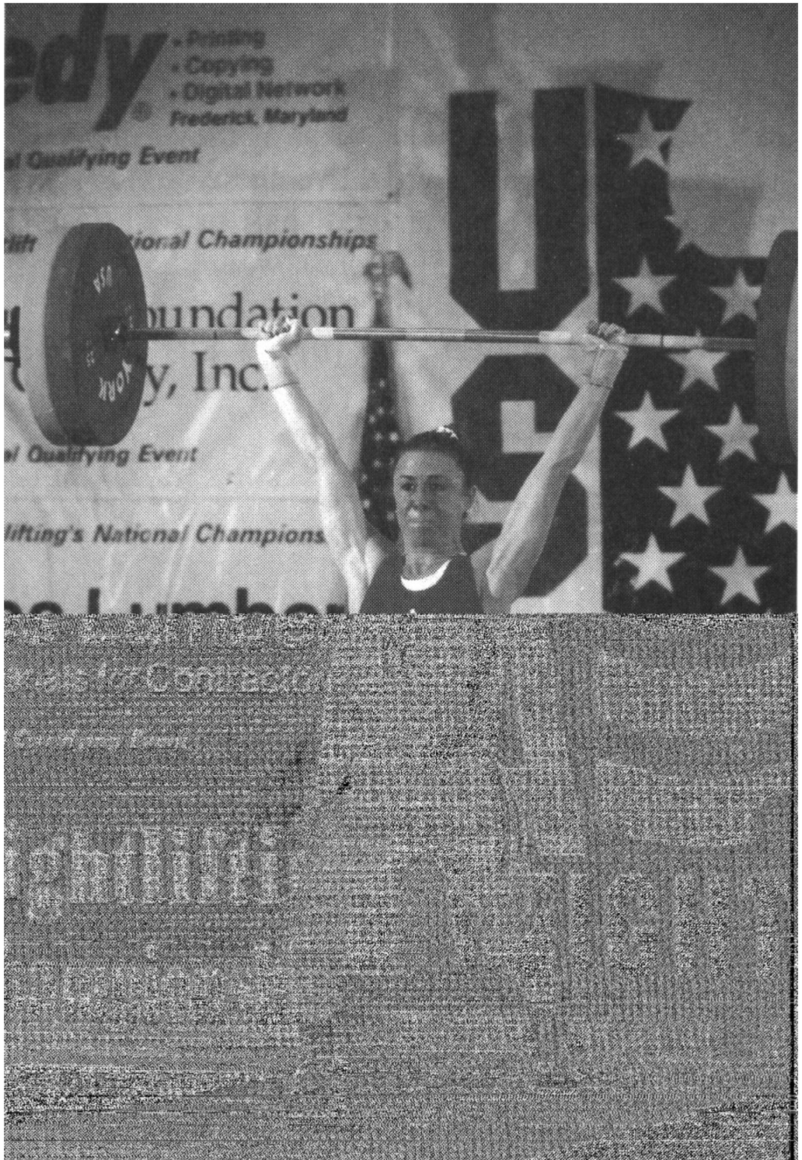
- ✓ **It depends on the beam cross-section**
- ✓ **We need some particular properties of the section**

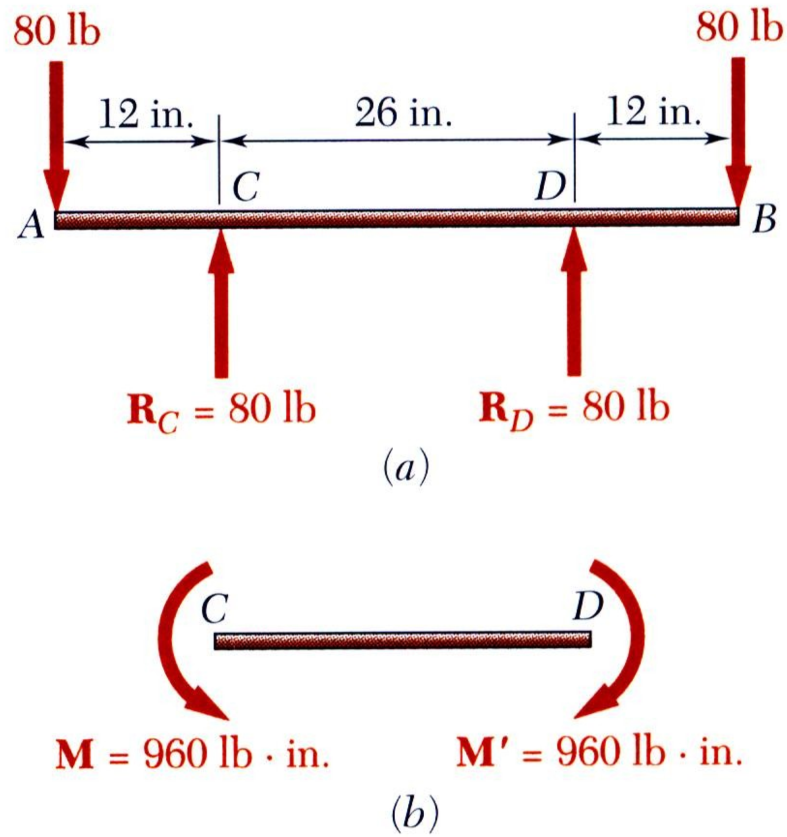


how big & what shape?

is the section we are using as a beam

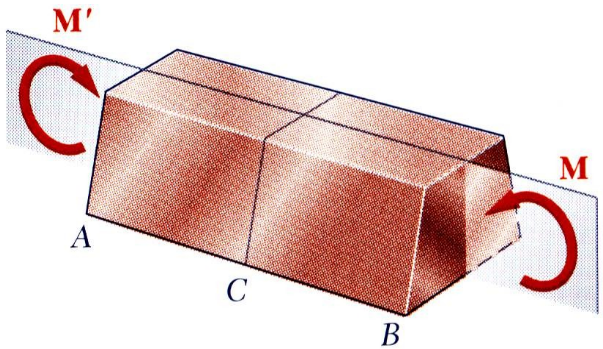
Pure Bending

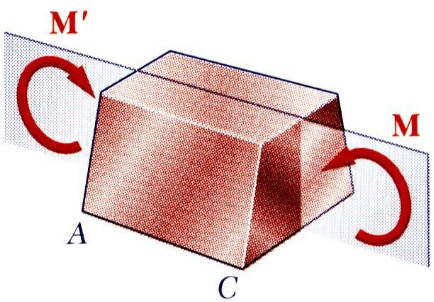


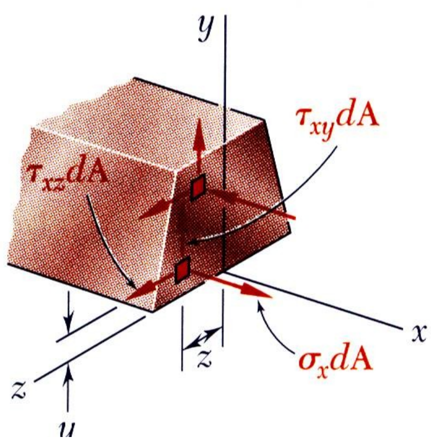


Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

Symmetric Member in Pure Bending







- ✓ Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section ***bending moment***.
- ✓ From statics, a couple M consists of two equal and opposite forces.
- ✓ The sum of the components of the forces in any direction is zero.
- ✓ The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.

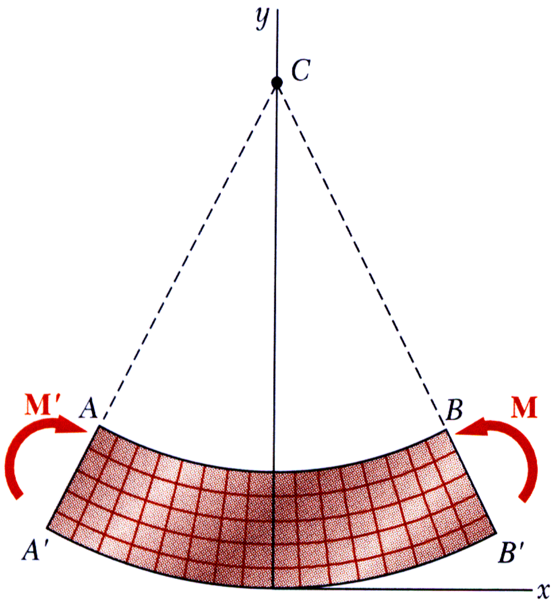
$$F_x = \int \sigma_x \, dA = 0$$

$$M_y = \int z\sigma_x \, dA = 0$$

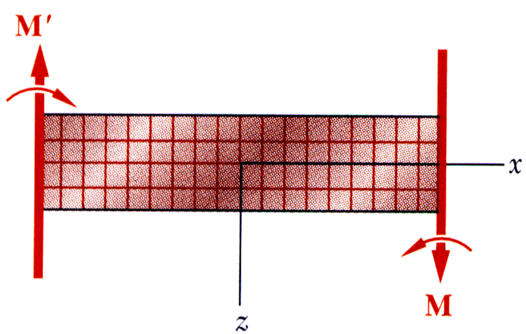
$$M_z = \int -y\sigma_x \, dA = M$$

- ✓ These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

Bending Deformations



(a) Longitudinal, vertical section (plane of symmetry)

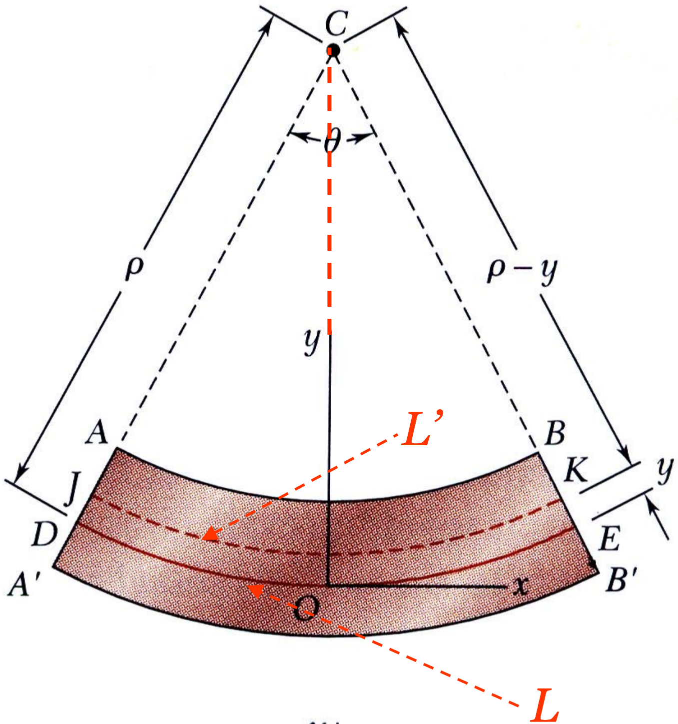


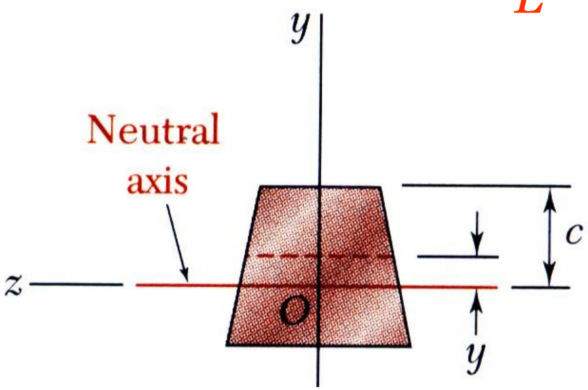
(b) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- ✓ member remains symmetric
- ✓ bends uniformly to form a circular arc
- ✓ cross-sectional plane passes through arc center and remains planar
- ✓ length of top decreases and length of bottom increases
- ✓ a ***neutral surface*** must exist that is parallel to the upper and lower surfaces and for which the length does not change
- ✓ stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending





Consider a beam segment of length L.

After deformation, the length of the neutral surface remains L. At other sections,

$$L' = (\rho - y)\theta$$

$$\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} = -\kappa y \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

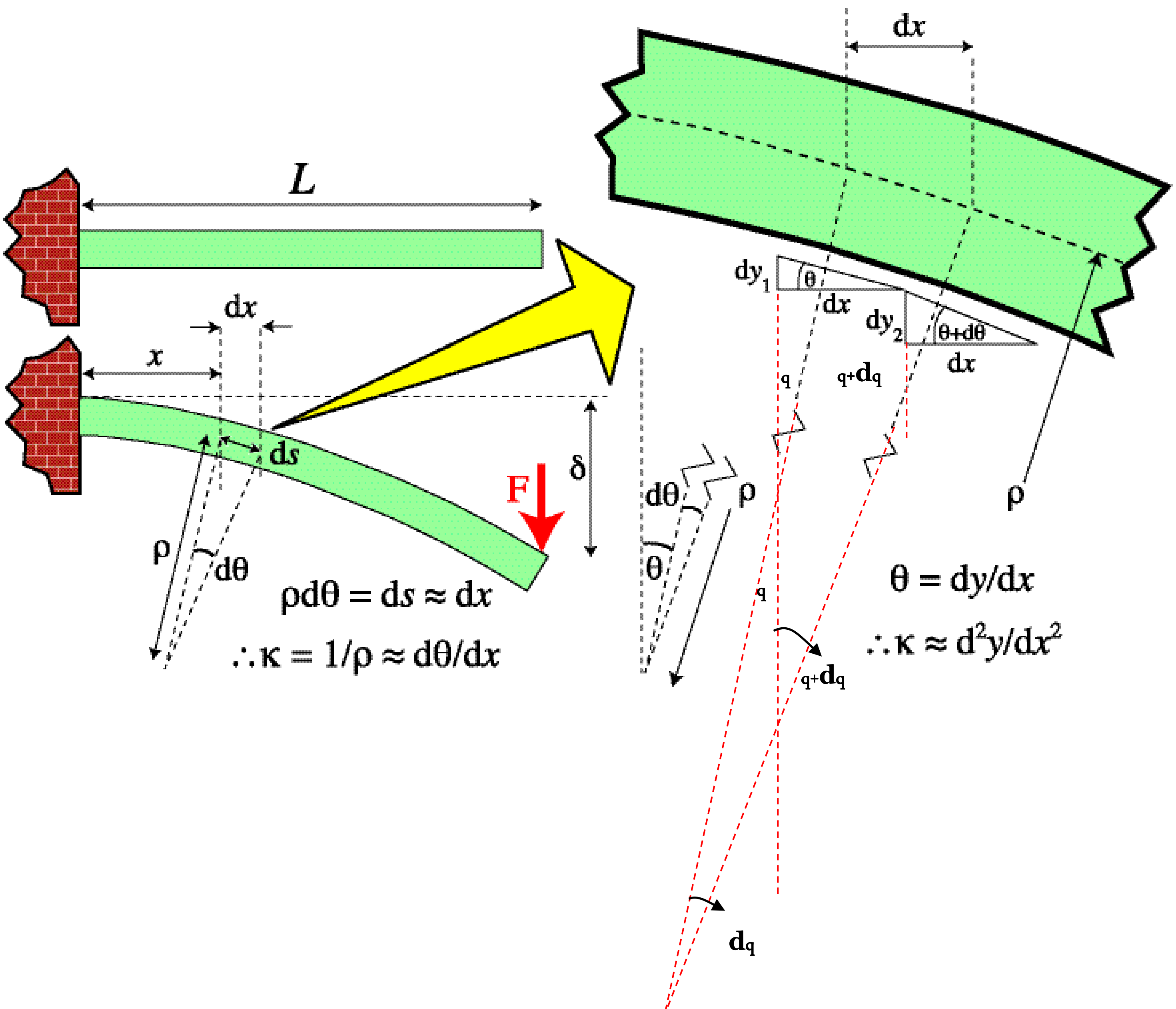
maximum strain in a cross section

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

$e_x < 0 \Rightarrow$ shortening $\Rightarrow$ compression (y>0, k <0)

$e_x > 0 \Rightarrow$ elongation $\Rightarrow$ tension (y<0, k >0)

Curvature



A small radius of curvature, ρ, implies large curvature of the beam, κ, and vice versa. In most cases of interest, the curvature is small, and we can approximate $ds \cong dx$.

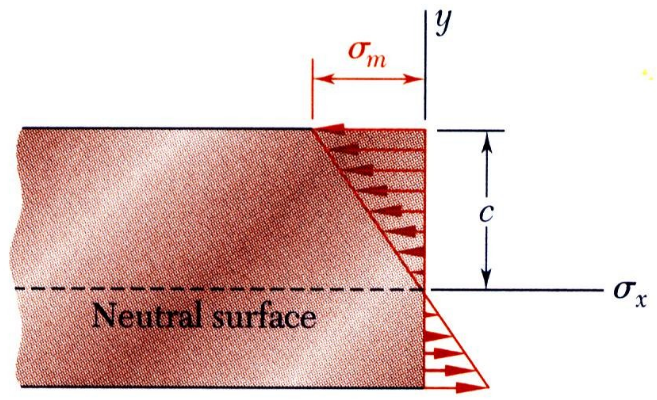
Stress Due to Bending

✓ For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{\rho}E = -\frac{y}{c}E\varepsilon_m$$

$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

maximum stress in a cross section

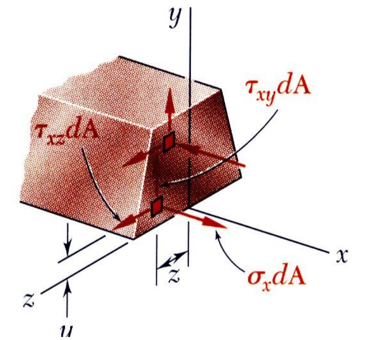


✓ For static equilibrium,

$$F_x = 0 = \int_A \sigma_x \, dA = \int_A \left(-E\frac{y}{\rho}\right) dA = \int_A (-E\kappa y) \, dA$$

$$0 = -\frac{E}{\rho}\int_A y \, dA \Rightarrow$$

$$\Rightarrow \int_A y \, dA = 0$$



First moment with respect to neutral plane (*z*-axis) is zero. Therefore, the neutral surface must pass through the section centroid.

Moment-curvature relationship

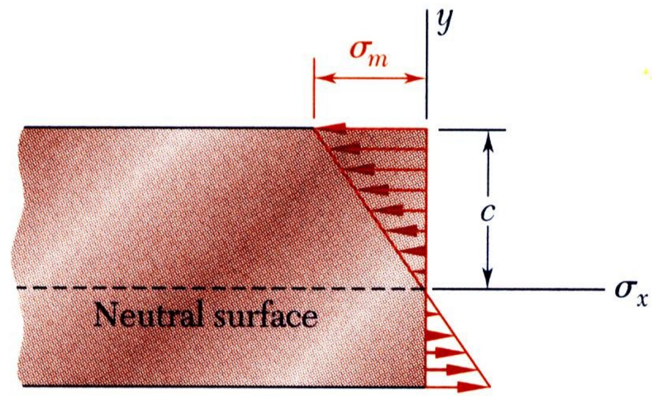
✓ The moment of the resultant of the stresses dF about the N.A.:

$$M = \int_A -y\sigma_x\, dA = \int_A -y\left(-\frac{y}{c}\sigma_m\right) dA$$

$$M = \frac{\sigma_m}{c}\int y^2\, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Substituting $\sigma_x = -\frac{y}{c}\sigma_m \Rightarrow \sigma_x = -\frac{My}{I}$



$$I = \int_A y^2\, dA$$

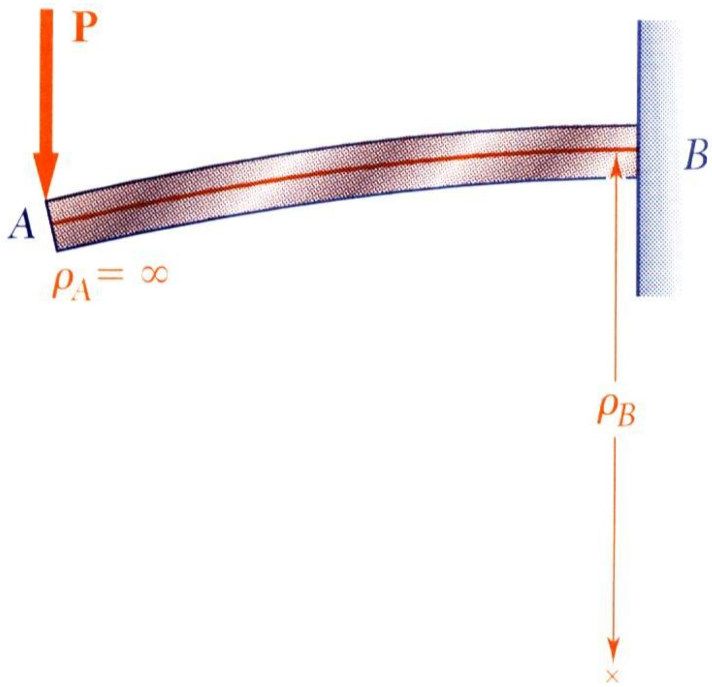
is the ***'second moment of area'***

$$M = \int_A -y\sigma_x\, dA = \int_A -y(-\kappa E y)\, dA$$

$$M = \kappa E \int_A y^2\, dA = \kappa E I$$

$$\kappa = \frac{M}{EI},$$

Deformation of a Beam Under Transverse Loading



- Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\kappa = \frac{1}{\rho} = \frac{M(x)}{EI}$$

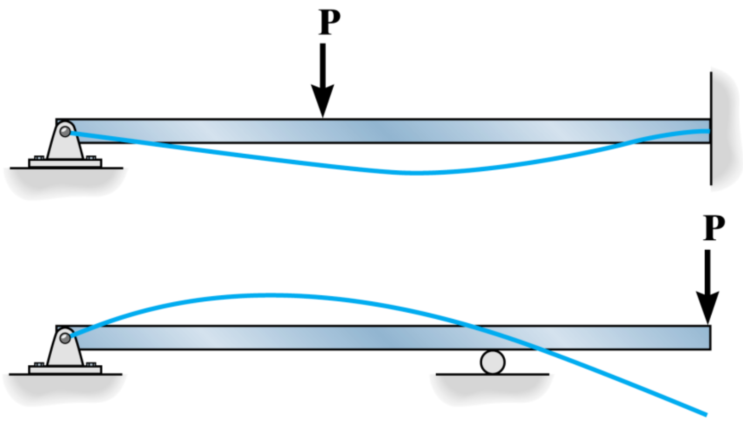
- Cantilever beam subjected to concentrated load at the free end,

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

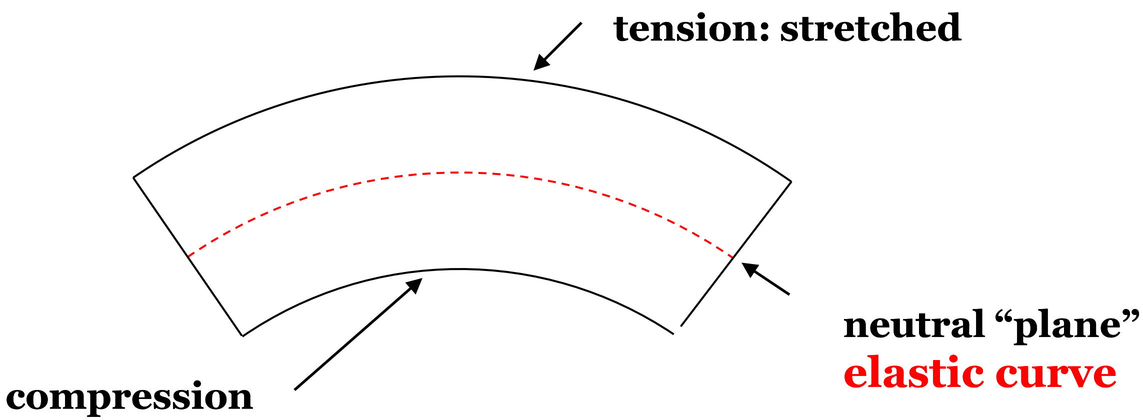
- At the free end A, $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$

- At the support B, $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

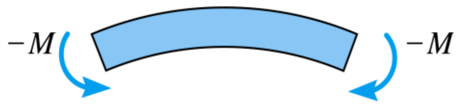
Elastic Curve



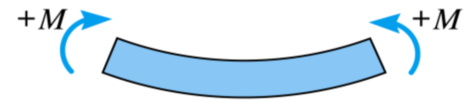
The deflection diagram of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam is called the elastic curve, which is characterized by the deflection and slope along the curve.



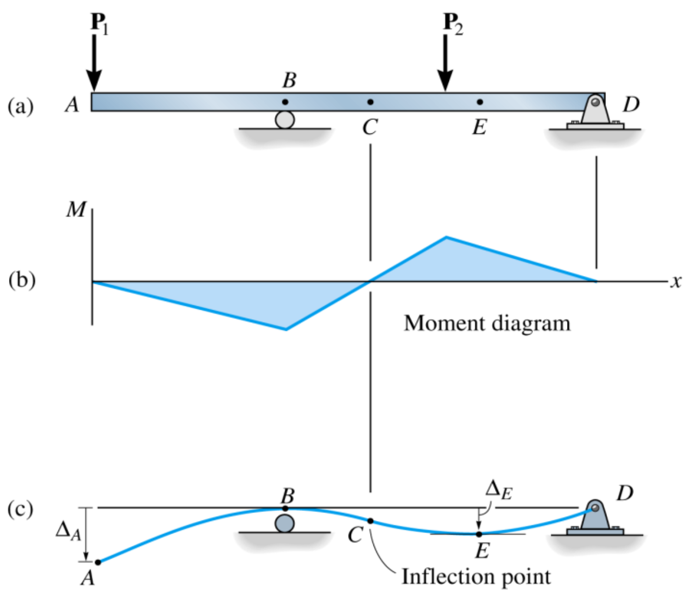
Moment-curvature relationship: Sign convention



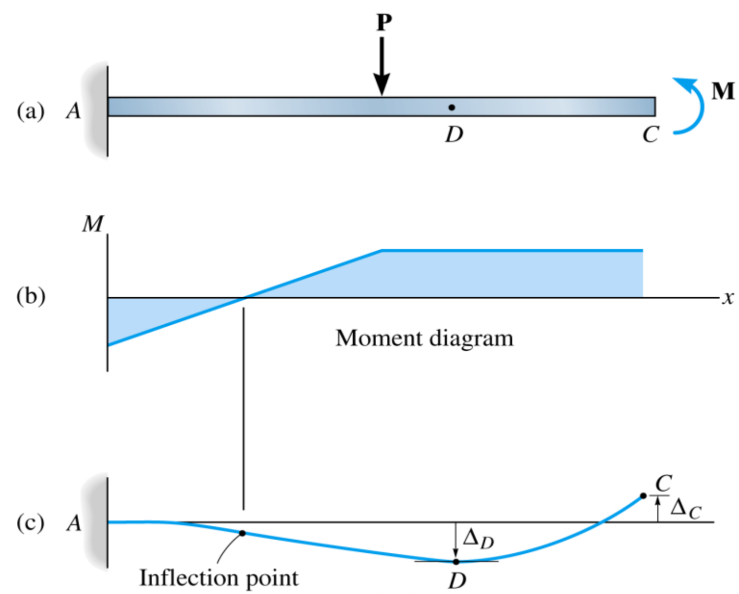
Negative internal moment
concave downwards
(b)



Positive internal moment
concave upwards
(a)



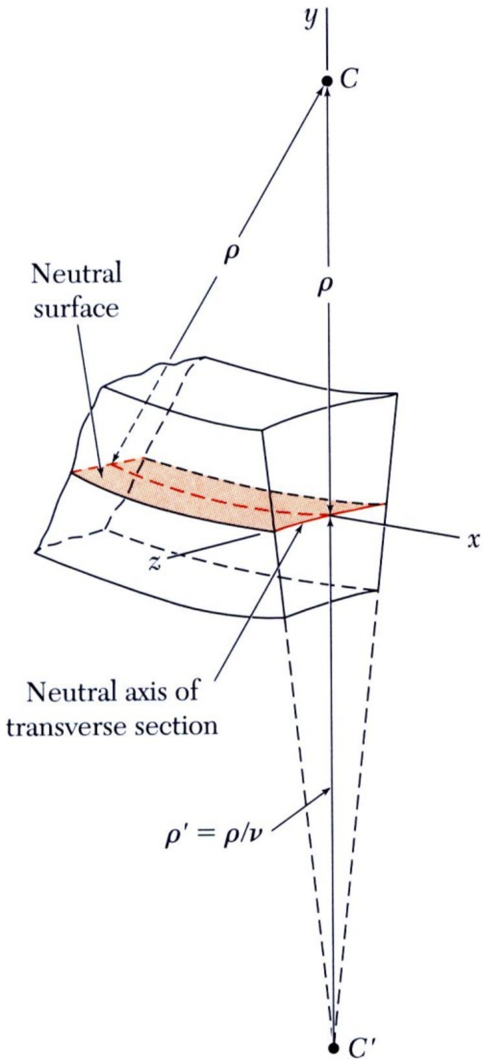
Elastic curve



Elastic curve

Maximum curvature occurs where the moment magnitude is a maximum.

Deformations in a Transverse Cross Section



✓ Deformation due to bending moment M is quantified by the curvature of the neutral surface

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

$$= \frac{M}{EI}$$

✓ Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$

✓ Expansion above the neutral surface and contraction below it causes an in-plane curvature,

$$\frac{1}{\rho'} = \frac{\nu}{\rho} = \text{anticlastic curvatur}$$