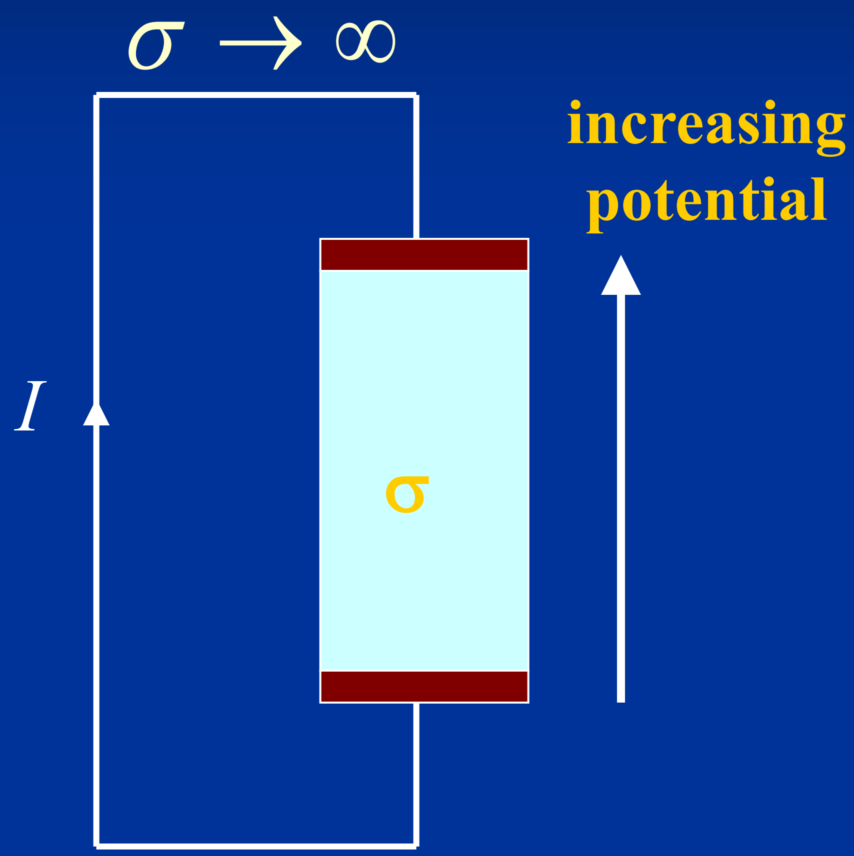
Electromotive Force & Continuity Equation

Electromotive Force

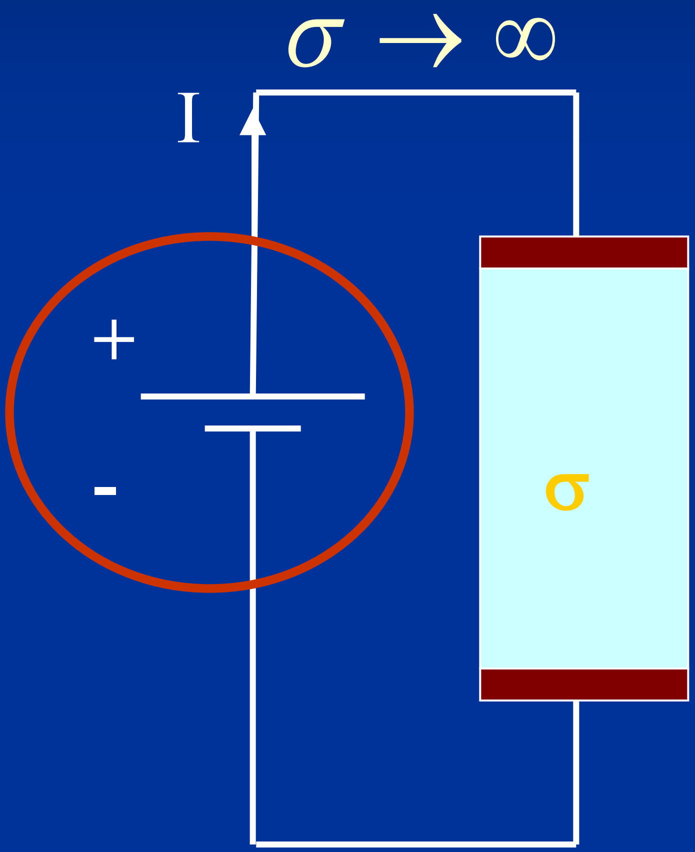
- Steady current flow requires a closed circuit.
- Electrostatic fields produced by stationary charges are conservative. Thus, they cannot by themselves maintain a steady current flow.

Electromotive Force (Cont'd)



- The current I must be zero since the electrons cannot gain back the energy they lose in traveling through the resistor.

Electromotive Force (Cont’d)

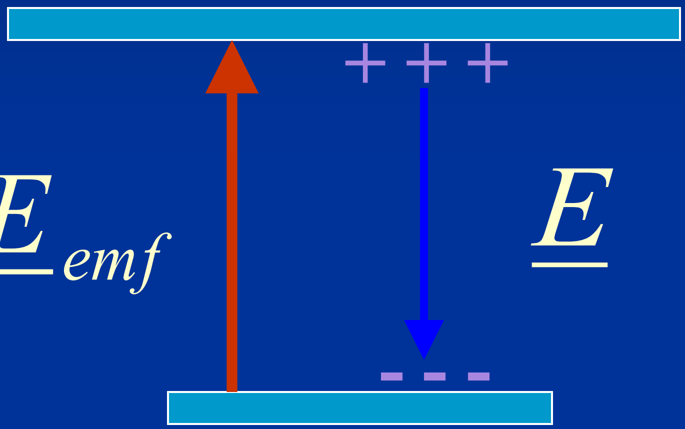


- To maintain a steady current, there must be an element in the circuit wherein the potential ***rises*** along the direction of the current.

Electromotive Force (Cont'd)

- For the potential to rise along the direction of the current, there must be a ***source*** which converts some form of energy to electrical energy.
- Examples of such sources are:
  - **batteries**
  - **generators**
  - **thermocouples**
  - **photo-voltaic cells**

Inside the Voltage Source



$\underline{E}_{emf}$ $\underline{E}$

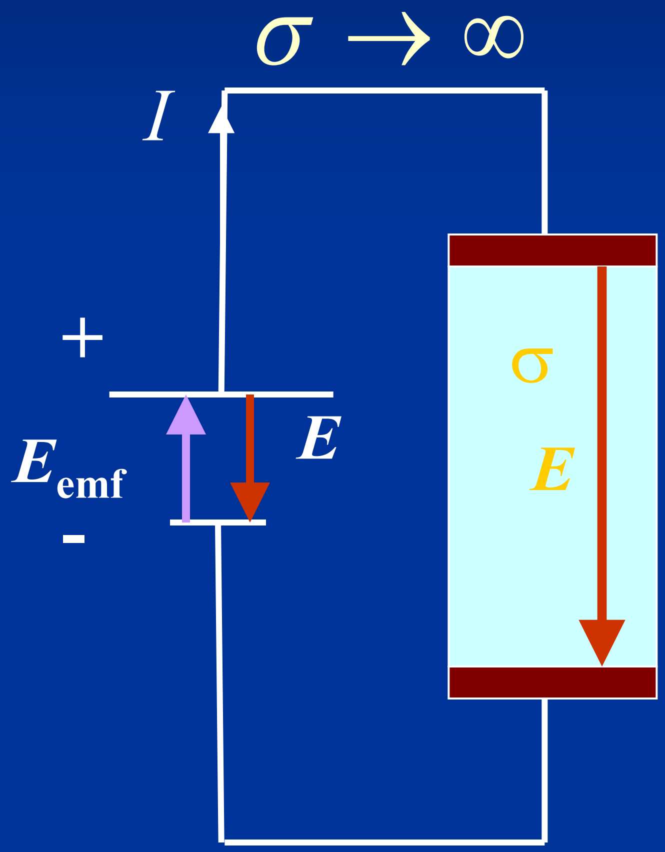
- E_{emf} is the electric field established by the energy conversion.
- This field moves positive charge to the upper plate, and negative charge to the lower plate.
- These charges establish an electrostatic field E.

In equilibrium:

$$\underline{E}_{emf} + \underline{E} = 0$$

Source is not connected to external world.

Electromotive Force (Cont'd)



At all points in the circuit, we must have

$$\frac{\underline{J}}{\sigma} = \underline{E}_{total} = \underline{E}_{emf} + \underline{E}$$

exists only in battery

Lecture 6

Electromotive Force (Cont'd)

- Integrate around the circuit in the direction of current flow

$$\oint_C \underline{E}_{total} \cdot d\underline{l} = \oint_C \frac{1}{\sigma} \underline{J} \cdot d\underline{l}$$

$$\underbrace{\oint_C \underline{E} \cdot d\underline{l}}_{0} + \int_{-}^{+} \underline{E}_{emf} \cdot d\underline{l} = \oint_C \frac{1}{\sigma} \underline{J} \cdot d\underline{l}$$

Electromotive Force (Cont'd)

- Define the ***electromotive force*** (*emf*) or "voltage" of the battery as

$$V_{emf} = \int_{-}^{+} \underline{E}_{emf} \cdot d\underline{l}$$

Electromotive Force (Cont'd)

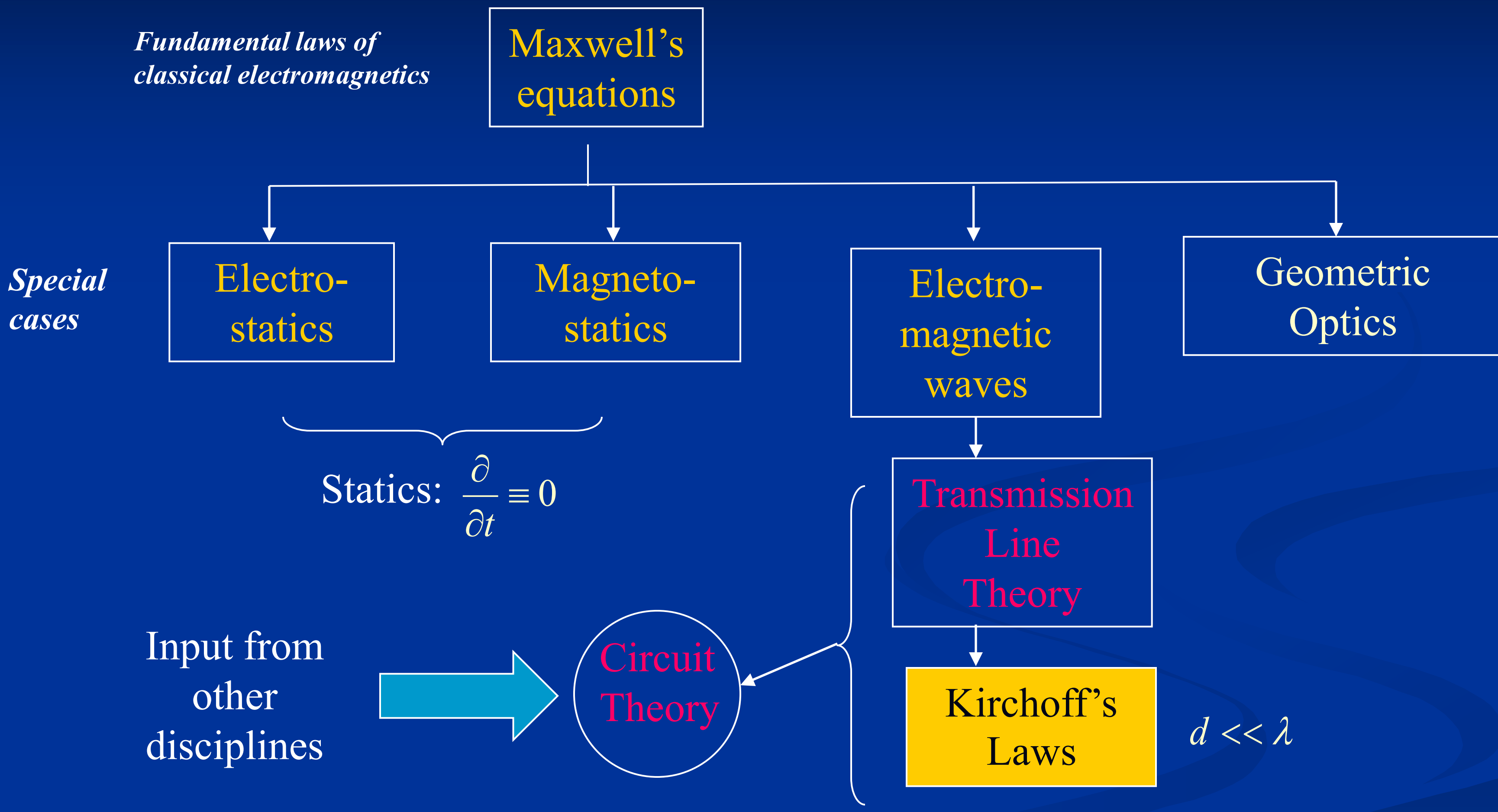
- We also note that

$$\oint_C \frac{1}{\sigma} \underline{J} \cdot d\underline{l} = \frac{l}{\sigma A} I = RI$$

- Thus, we have the circuit relation

$$V_{emf} = RI$$

Overview of Electromagnetics



Kirchhoff's Voltage Law

- For a closed circuit containing voltages sources and resistors, we have

$$\boxed{\sum V_{emf} = I \sum R}$$

• "the algebraic sum of the *emf*s around a closed circuit equals the algebraic sum of the voltage drops over the resistances around the circuit."

Kirchhoff's Voltage Law

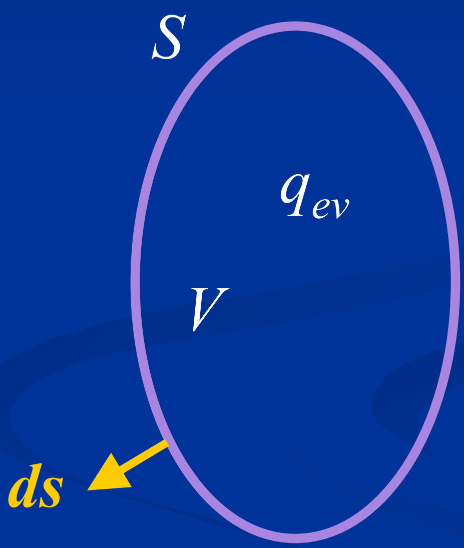
- Strictly speaking KVL only applies to circuits with steady currents (DC).
- However, for AC circuits having dimensions much smaller than a wavelength, KVL is also approximately applicable.

Conservation of Charge

- Electric charges can neither be created nor destroyed.
- Since current is the flow of charge and charge is conserved, there must be a relationship between the current flow out of a region and the rate of change of the charge contained within the region.

Conservation of Charge (Cont'd)

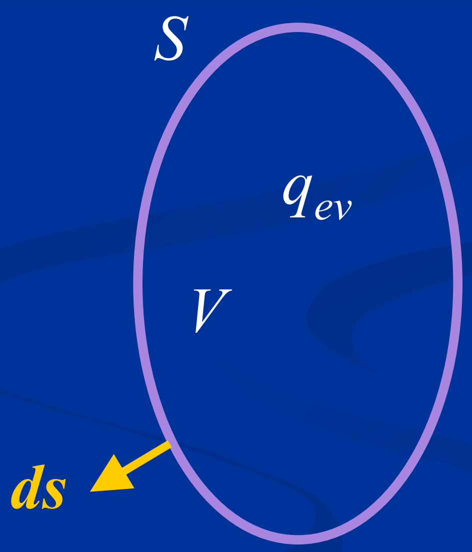
- Consider a volume V bounded by a closed surface S in a homogeneous medium of permittivity ε and conductivity σ containing charge density q_{ev}.



Conservation of Charge (Cont'd)

- The net current leaving V through S must be equal to the time rate of decrease of the total charge within V, i.e.,

$$I = -\frac{dQ_{enc}}{dt}$$



Conservation of Charge (Cont'd)

- The net current leaving the region is given by

$$I = \oint_S \underline{J} \cdot d\,\underline{s}$$

- The total charge enclosed within the region is given by

$$Q = \int_V q_{ev}\, dv$$

Conservation of Charge (Cont'd)

- Hence, we have

$$\boxed{\oint_S \underline{J} \cdot d\,\underline{s} = -\frac{d}{dt}\int_V q_{ev}\, dv}$$

net outflow of current

net rate of decrease of total charge

Continuity Equation

- Using the *divergence theorem*, we have

$$\oint_S \underline{J} \cdot d\,\underline{s} = \int_V \nabla \cdot \underline{J}\, dv$$

- We also have

$$\frac{d}{dt}\int_V q_{ev}\, dv = \int_V \frac{\partial q_{ev}}{\partial t}\, dv$$

Becomes a partial derivative when moved inside of the integral because q_{ev} is a function of position as well as time.

Continuity Equation (Cont'd)

- Thus,

$$\int_V \nabla \cdot \underline{J}\, dv + \int_V \frac{\partial \rho}{\partial t}\, dv = 0$$

- Since the above relation must be true for *any and all regions*, we have

$$\boxed{\nabla \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0}$$

Continuity Equation

Continuity Equation (Cont'd)

- For steady currents, $$\frac{\partial \rho}{\partial t} = 0$$

- Thus,

$$\boxed{\nabla \cdot \underline{J} = 0}$$

J is a *solenoidal* vector field.

Continuity Equation in Terms of Electric Field

- Ohm's law in a conducting medium states

$$\underline{J} = \sigma \underline{E}$$

- For a homogeneous medium

$$\nabla \cdot \underline{J} = \sigma \nabla \cdot \underline{E} = 0 \quad \Rightarrow \quad \nabla \cdot \underline{E} = 0$$

- But from Gauss's law,

$$\nabla \cdot \underline{E} = \frac{q_{ev}}{\varepsilon}$$

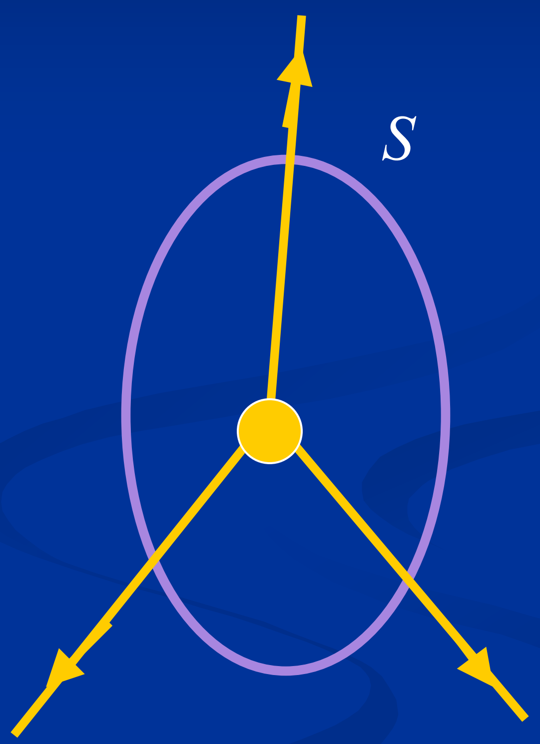
- Therefore, the volume charge density, ρ, must be zero in a homogeneous conducting medium

Kirchhoff's Current Law

- Since ***J*** is solenoidal, we must have

$$\oint_S \underline{J} \cdot d\,\underline{s} = 0$$

- In a circuit, steady current flows in wires.
- Consider a "node" in a circuit.



Kirchoff's Current Law (Cont'd)

- We have for a node in a circuit

$$\boxed{\sum I = 0}$$

- **"the algebraic sum of all currents leaving a junction must be zero."**

Kirchoff's Current Law (Cont'd)

- Strictly speaking KCL only applies to circuits with steady currents (DC).
- However, for AC circuits having dimensions much smaller than a wavelength, KCL is also approximately applicable.

Redistribution of Free Charge

- Charges introduced into the interior of an isolated conductor migrate to the conductor surface and redistribute themselves in such a way that the following conditions are met:
  - $\boldsymbol{E} = 0$ within the conductor
  - $\boldsymbol{E}_t = 0$ just outside the conductor
  - $q_{ev} = 0$ within the conductor
  - $q_{es} \neq 0$ on the surface of the conductor

Redistribution of Free Charge (Cont'd)

- We can derive the differential equation governing the redistribution of charge from Gauss's law in differential form and the continuity equation.
- From Gauss's law for the electric field, we have

$$\nabla \cdot \underline{D} = q_{ev} \quad \Rightarrow \quad \nabla \cdot \underline{E} = \frac{q_{ev}}{\varepsilon} \quad \Rightarrow \quad \nabla \cdot \underline{J} = \frac{\sigma}{\varepsilon} q_{ev}$$

Redistribution of Free Charge (Cont'd)

- From the continuity equation, we have

$$\nabla \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

- Combining the two equations, we obtain

$$\boxed{\frac{\partial \rho(\underline{r},t)}{\partial t} + \frac{\sigma}{\varepsilon}\rho(\underline{r},t) = 0}$$

Describes the time evolution of the charge density at a given location.

Redistribution of Free Charge (Cont'd)

- The solution to the DE is given by

$$\boxed{\rho(\underline{r},t) = \rho_0(\underline{r})\, e^{-(t/\tau_r)}}$$

Initial charge distribution at $t = 0$

where $\tau_r = \varepsilon/\sigma$ is the time constant of the process called the ***relaxation time***.

Redistribution of Free Charge (Cont'd)

- The initial charge distribution at any point in the bulk of the conductor decays exponentially to zero with a time constant τ_r.
- At the same time, surface charge is building up on the surface of the conductor.
- The *relaxation time* decreases with increasing conductivity.
- For a good conductor, the time required for the charge to decay to zero at any point in the bulk of the conductor (and to build up on the surface of the conductor) is very small.

Relaxation Times for Some Materials

$$\tau_r^{copper} = 1.5 \times 10^{-19}\,\mathrm{s}$$

$$\tau_r^{H_2O} \approx 10^{-5}\,\mathrm{s}$$

$$\tau_r^{amber} \approx 4 \times 10^{3}\,\mathrm{s}$$

$$\tau_r^{mica} \approx 10 \text{ to } 20 \text{ hrs}$$

$$\tau_r^{quartz} \approx 50 \text{ days}$$

Electrical Nature of Materials as a Function for Frequency

- The concept of ***relaxation time*** is also used to determine the electrical nature (conductor or insulator) of materials at a given frequency.
- A material is considered to be a good conductor if

$$\tau_r << T = \frac{1}{f} \quad \Rightarrow \quad \tau_r f << 1$$

Electrical Nature of Materials as a Function for Frequency (Cont'd)

- A material is considered to be a good insulator if

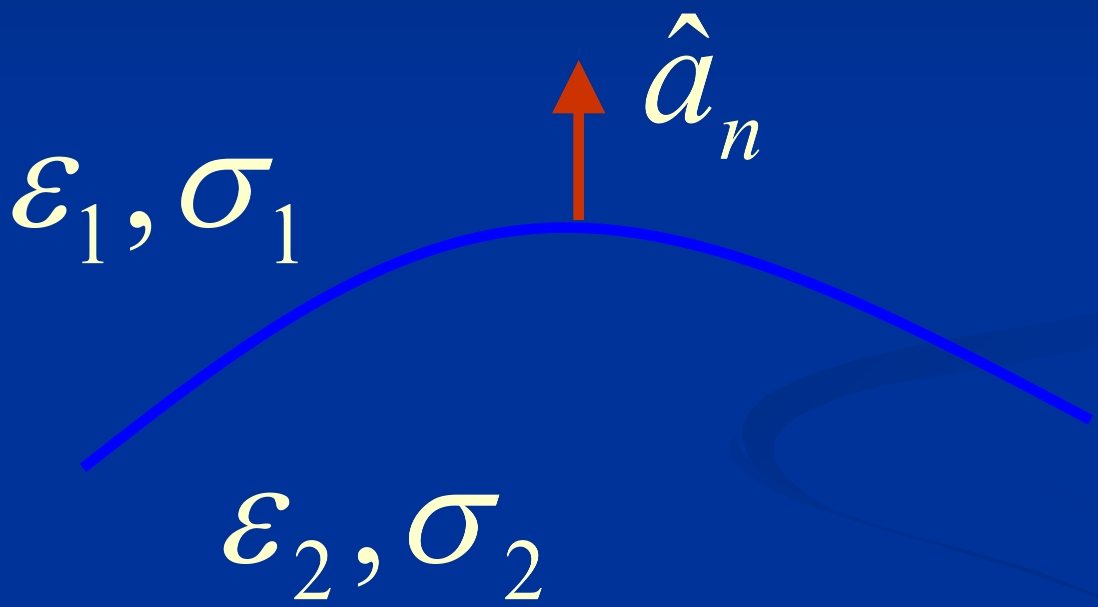
$$\tau_r >> T = \frac{1}{f} \quad \Rightarrow \quad \tau_r f >> 1$$

- A good conductor is a material with a relaxation time such that any free charges deposited within its bulk migrate to its surface long before a period of the wave has passed.

Boundary Conditions for Steady Current Flow

- The behavior of current flow across the interface between two different materials is governed by boundary conditions.
- The boundary conditions for current flow are obtained from the ***integral forms*** of the basic equations governing current flow.

Boundary Conditions for Steady Current Flow (Cont'd)



Boundary Conditions for Steady Current Flow (Cont'd)

- The governing equations for steady electric current (in a conductor) are:

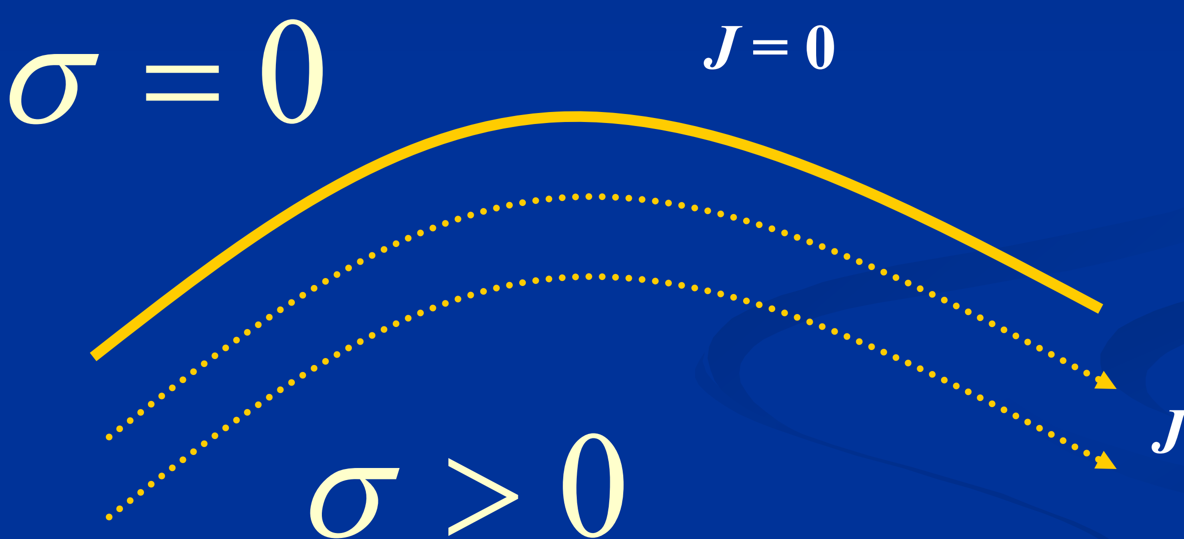
$$\oint_S \underline{J} \cdot d\underline{s} = 0$$

$$\oint_C \underline{E} \cdot d\underline{l} = 0 \quad \Rightarrow \oint_C \frac{\underline{J}}{\sigma} \cdot d\underline{l} = 0$$

Boundary Conditions for Steady Current Flow (Cont'd)

- The normal component of a ***solenoidal*** vector field is continuous across a material interface:
$$\boxed{J_{1n} = J_{2n}}$$
- The tangential component of a ***conservative*** vector field is continuous across a material interface:
$$\boxed{\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}}$$

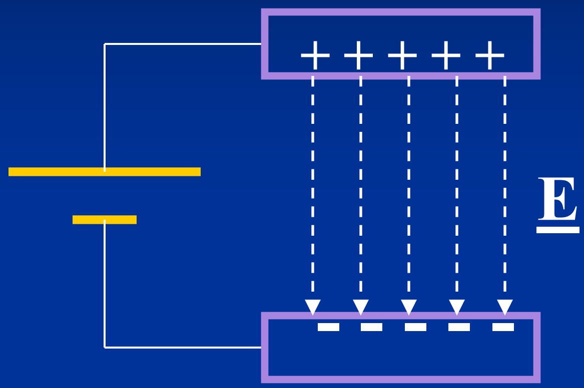
Conductor-Dielectric Interface



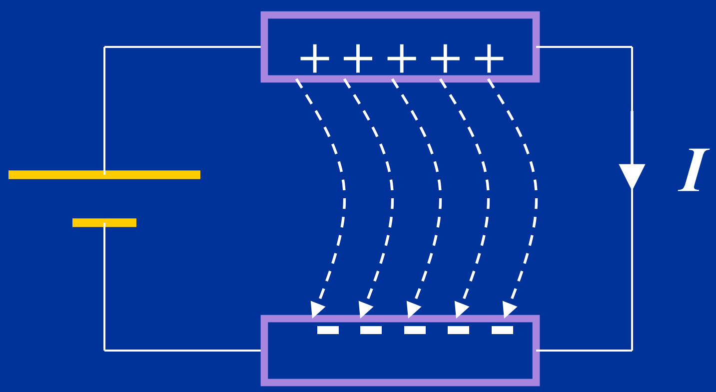
Conductor-Dielectric Interface (Cont'd)

- The current in the conductor must flow tangential to the boundary surface.
- The tangential component of the electric field must be continuous across the interface.
- The normal component of the electric field must be zero at the boundary inside the conductor, but not in the dielectric. Thus, there will be a buildup of surface charge at the interface.

Conductor-Dielectric Interface (Cont'd)



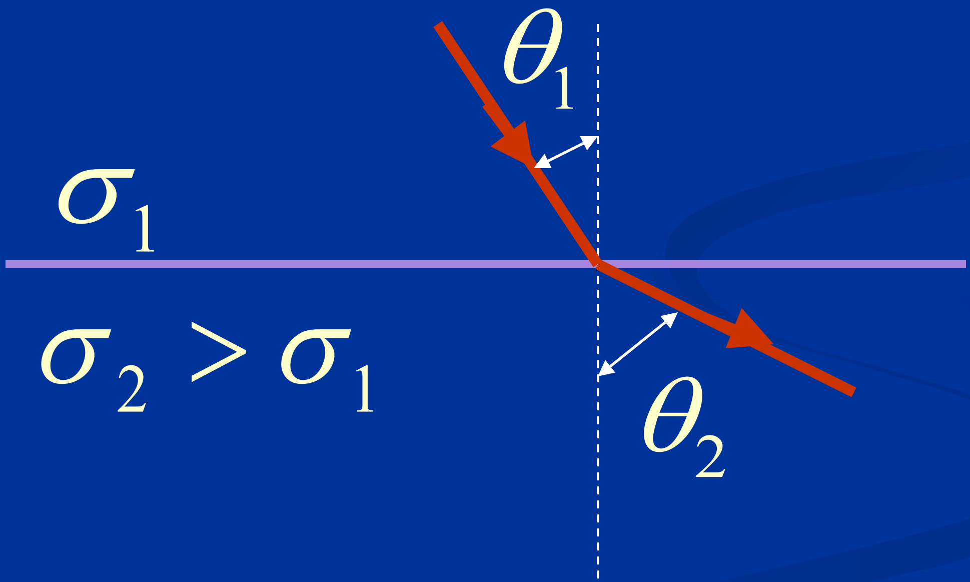
no current flow $\Rightarrow E_t = 0$



current flow $\Rightarrow E_n >> E_t$

Conductor-Conductor Interface

- The current bends as it cross the interface between two conductors



Conductor-Conductor Interface (Cont'd)

- The angles are related by $$\theta_2 = \tan^{-1}\left(\frac{\sigma_2}{\sigma_1}\tan\theta_1\right)$$
- Suppose medium 1 is a good conductor and medium 2 is a good insulator (i.e., $\sigma_1 >> \sigma_2$). Then $\theta_2 \approx 0$. In other words, the current enters medium 2 at nearly right angles to the boundary. This result is consistent with the fact that the electric field in medium 2 should have a vanishingly small tangential component at the interface.

Conductor-Conductor Interface (Cont'd)

- In general, there is a buildup of surface charge at the interface between two conductors.

$$J_{1n} = J_{2n} = J_n = \sigma_1 E_{1n} = \sigma_2 E_{2n}$$

$$\rho_s = D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}$$

$$= \left(\varepsilon_1 - \varepsilon_2 \frac{\sigma_1}{\sigma_2} \right) E_{1n} = J_n \left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2} \right)$$

$$= J_n \left(\tau_{r1} - \tau_{r2} \right)$$

$\Rightarrow$ Only when the relaxation times of the two conductors are equal is there no buildup of surface charge at the interface.