*Cauchy's Integral Theorem and Formula

Cauchy's integral Theorem

We now turn to integration.

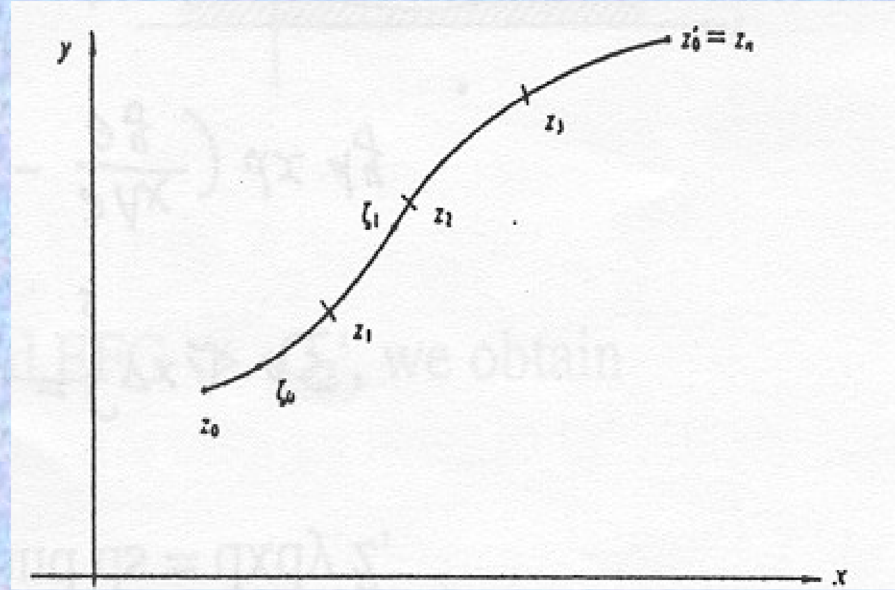
in close analogy to the integral of a real function

The contour $z_0 \to z_0^{'}$ is divided into n intervals .Let $n \to \infty$

with $|\Delta z_j| = |z_j - z_{j-1}| \to 0$ for j. Then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)\Delta z_j = \int_{z_0}^{z_0'} f(z)dz$$

provided that the limit exists and is independent of the details of choosing the points z_j and ζ_j, where ζ_j is a point on the curve bewteen z_j and z_{j-1}.



The right-hand side of the above equation is called the contour (path) integral of f(z)

As an alternative, the contour may be defined by

$$\int_{c\ z_1}^{z_2} f(z)dz = \int_{c\ x_1 y_1}^{x_2 y_2} [u(x,y) + iv(x,y)][dx + idy]$$

$$= \int_{c\ x_1 y_1}^{x_2 y_2} [udx - vdy] + i \int_{c\ x_1 y_1}^{x_2 y_2} [vdx + udy]$$

with the path C specified. This reduces the complex integral to the complex sum of real integrals. It's somewhat analogous to the case of the vector integral.

An important example $\int_c z^n dz$

where C is a circle of radius r>0 around the origin z=0 in the direction of counterclockwise.

In polar coordinates, we parameterize $z = re^{i\theta}$

and $dz = ire^{i\theta}d\theta$ and have

$$\frac{1}{2\pi i}\int_c z^n dz = \frac{r^{n+1}}{2\pi}\int_0^{2\pi} \exp[i(n+1)\theta]d\theta$$

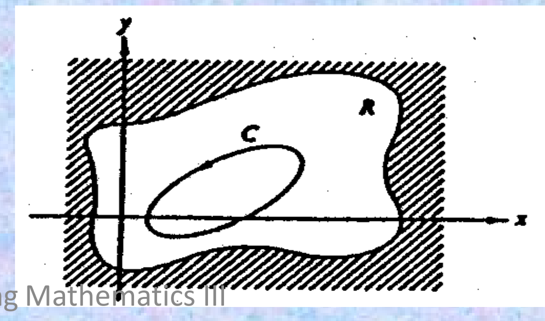
$$= \begin{cases} 0 & \text{for } n \neq -1 \\ 1 & \text{for } n = -1 \end{cases}$$

which is independent of r.

Cauchy's integral theorem

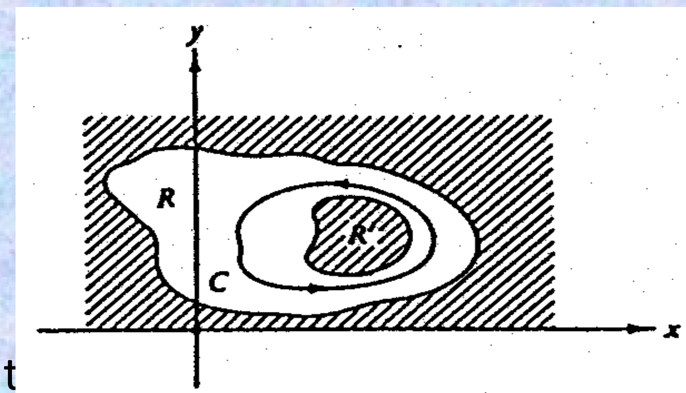
– If a function f(z) is analytical (therefore single-valued) [and its partial derivatives are continuous] through some simply connected region **R**, for every closed path C in **R**,

$$\oint_c f(z)dz = 0$$



• *Multiply connected regions*

The original statement of our theorem demanded a simply connected region. This restriction may easily be relaxed by the creation of a barrier, a contour line. Consider the multiply connected region of Fig.1.6 In which f(z) is not defined for the interior R′



1.6 Fig.

Cauchy's int … contour C, but we can construct a C′ for which the theorem holds. If line segments DE and GA arbitrarily close together, then

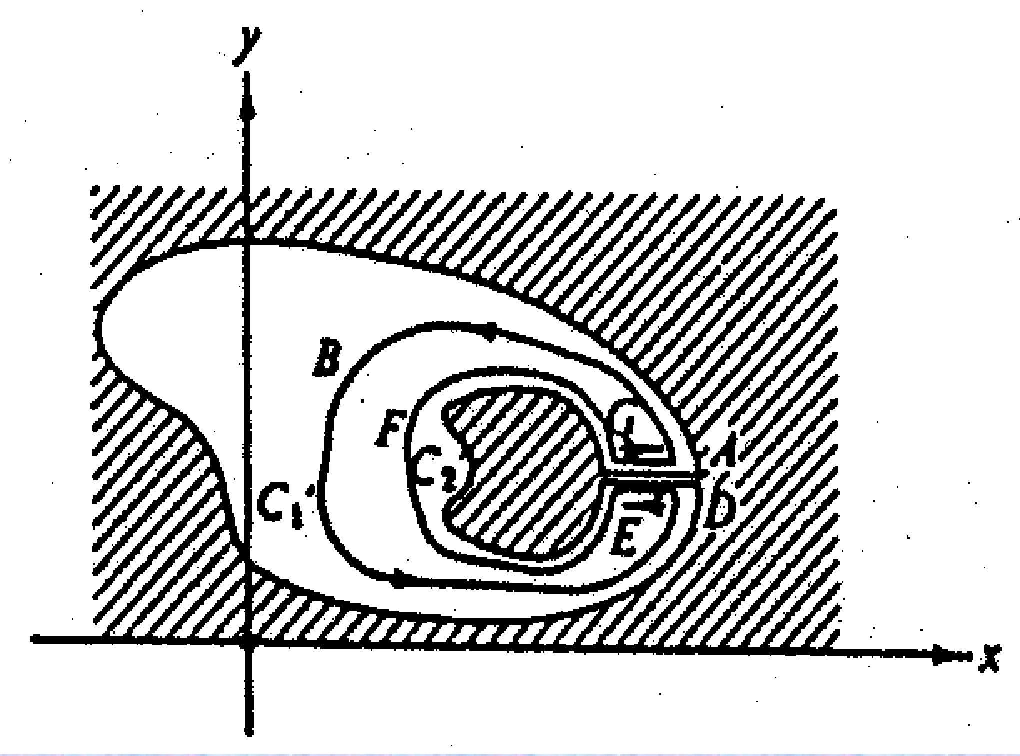
$$\int_{G}^{A} f(z)dz = -\int_{D}^{E} f(z)dz$$

$$\oint_{\substack{C' \\ (ABDEFGA)}} f(z)dz = \left[\int_{ABD} + \int_{DE} + \int_{GA} + \int_{EFG} \right] f(z)dz$$

$$= \left[\int_{\mathrm{ABD}} + \int_{\mathrm{EFG}} \right] \mathrm{f}(\mathrm{z})\mathrm{dz} = 0$$

$$\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz$$

$$ABD \to C_1' \quad EFG \to -C_2'$$



Cauchy's Integral Formula

Cauchy's integral formula: If f(z) is analytic on and within a closed contour C then

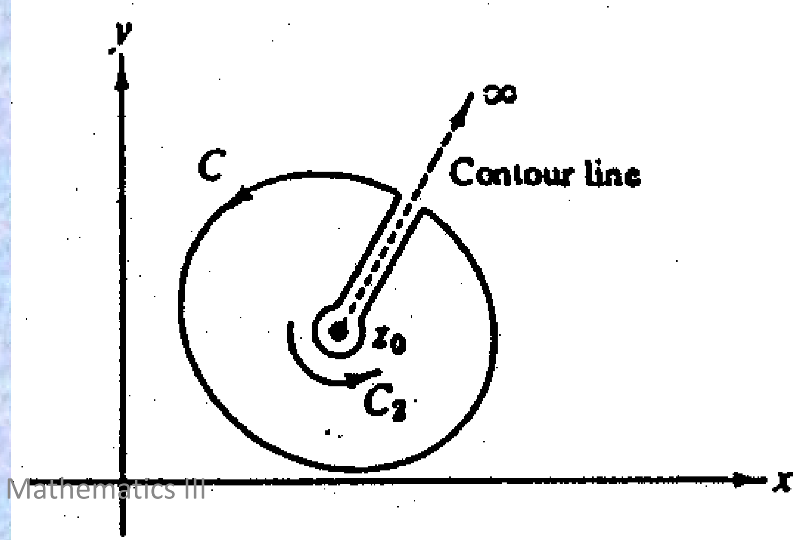
$$\oint_C \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0)$$

in which z_0 is some point in the interior region bounded by C. Note that here $z-z_0 \neq 0$ and the integral is well defined.

Although f(z) is assumed analytic, the integrand $(f(z)/z-z_0)$ is not analytic at $z=z_0$ unless $f(z_0)=0$. If the contour is deformed as in Fig.1.8 Cauchy's integral theorem applies.

So we have

$$\oint_C \frac{f(z)dz}{z-z_0} - \oint_{C_2} \frac{f(z)}{z-z_0}dz = 0$$



Let $z - z_0 = re^{i\theta}$, here r is small and will eventually be made to approach zero

$$\oint_{C_2} \frac{f(z)dz}{z - z_0} dz = \oint_{C_2} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta$$

(r→0) $$= if(z_0)\oint_{C_2} d\theta = 2\pi if(z_0)$$

Here is a remarkable result. The value of an analytic function is given at an interior point at z=z_0 once the values on the boundary C are specified.

What happens if z_0 is exterior to C?
In this case the entire integral is analytic on and within C, so the integral vanishes.

$$\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z - z_0} = \begin{cases} f(z_0), & z_0 \text{ interior} \\ 0, & z_0 \text{ exterior} \end{cases}$$

Derivatives

Cauchy’s integral formula may be used to obtain an expression for the derivation of f(z)

$$f'(z_0) = \frac{d}{dz_0}\left(\frac{1}{2\pi i}\oint \frac{f(z)dz}{z - z_0}\right)$$

$$= \frac{1}{2\pi i}\oint f(z)dz \frac{d}{dz_0}\left(\frac{1}{z - z_0}\right) = \frac{1}{2\pi i}\oint \frac{f(z)dz}{(z - z_0)^2}$$

Moreover, for the n-th order of derivative

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)dz}{(z - z_0)^{n+1}}$$

We now see that, the requirement that f(z) be analytic not only guarantees a first derivative but derivatives of all orders as well! The derivatives of f(z) are automatically analytic. Here, it is worth to indicate that the converse of Cauchy’s integral theorem holds as well

Examples

If $f(z)=\sum_{n\geq 0} \mathrm{a}_{\mathrm{n}} z^{n}$ is analytic on and within a circle about the origin, find a_n.

$$f^{(j)}(z)=j!a_j+\sum_{n-j\geq 1} a_n\{\ \}z^{n-j}$$

$$f^{(j)}(0)=j!a_j$$

$$a_n=\frac{f^{(n)}(0)}{n!}=\frac{1}{2\pi i}\oint\frac{f(z)dz}{z^{n+1}}$$