Cauchy-Riemann Equation

Functions of a complex variable

Let S be a set f of complex numbers.

A function defined on S is a rule that assigns to each *z* in S a complex number *w*.

$$\text{value of } f \text{ at } z, \text{ or } f(z)$$

or

$$w = f\left(z\right)$$

S is the domain of definition of f

$$w = \frac{1}{z}$$

sometimes refer to the function f itself, for simplicity.

$$w = z^2 + 1$$

Both a domain of definition and a rule are needed in order for a function to be well defined.

Suppose $w = u + iv$ is the value of a function f at $z = x + iy$

$$u + iv = f\left(x + iy\right)$$

$$or \quad f\left(z\right) = u\left(x, y\right) + iv\left(x, y\right)$$

real-valued functions of real variables x, y

or $f\left(z\right) = u\left(r, \theta\right) + iv\left(r, \theta\right)$

Ex.

$$f\left(z\right) = z^2$$

$$f\left(x + iy\right) = x^2 - y^2 + i2xy$$

$$u(x, y) = x^2 - y^2, \qquad v\left(x, y\right) = 2xy$$

$$f\left(re^{i\theta}\right) = r^2 \cos 2\theta + ir^2 \sin 2\theta$$

$$u\left(r, \theta\right) = r^2 \cos 2\theta \qquad v\left(r, \theta\right) = r^2 \sin 2\theta$$

when v=0

$f\left(Z\right)$ is a real-valued function of a complex variable.

$f(z) = P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ is a polynomial of degree n.

$\dfrac{P(z)}{Q(z)}$: rational function, defined when $Q(z) \neq 0$

For multiple-valued functions : usually assign one to get single-valued function

Ex. $z = re^{i\theta}, \quad z \neq 0$

$z^{\frac{1}{2}} = \pm\sqrt{r}e^{i\theta/2}, \quad -\pi < \theta \leq \pi \quad n\text{th} \quad \text{root}$

If we choose $f(z) = \sqrt{r}e^{i\theta/2} \quad (r > 0, -\pi < \theta < \pi)$

$$-\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2}$$

and $f(0) = 0$,

then f is well defined on the entrie complex plane except the ray $\theta = \pi$.

Mappings

$w = f(z)$ is not easy to graph as real functions are.

One can display some information about the function by indicating pairs of corresponding points z=(x,y) and w=(u,v). (draw z and w planes separately).

When a function f is thought of in this way. it is often refried to as a <u>mapping</u>, or <u>transformation</u>.

inverse image of w

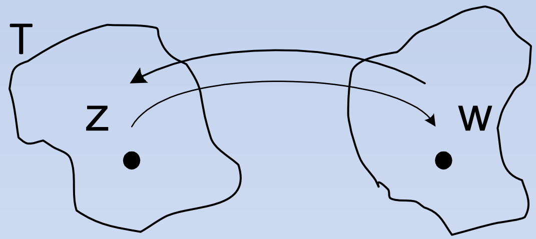

image of z

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Mapping can be translation, rotation, reflection. In such cases

it is convenient to consider *z* and *w* planes to be the same.

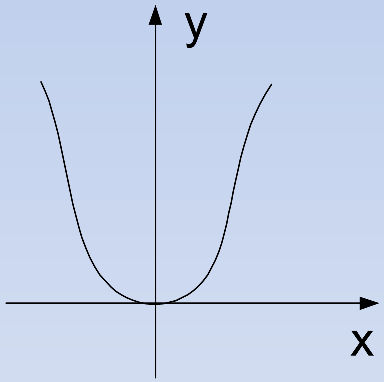
w=z+1 translation +1

w=iz rotation $\frac{\theta}{2}$

w= $\bar{z}$ reflection in real axis.

Ex. image of curves

real number $y = x^2$

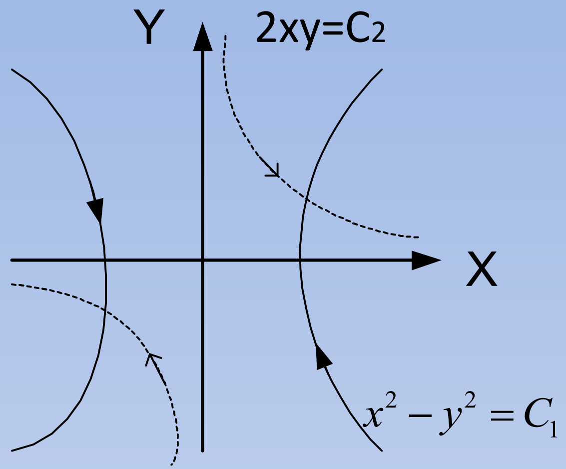


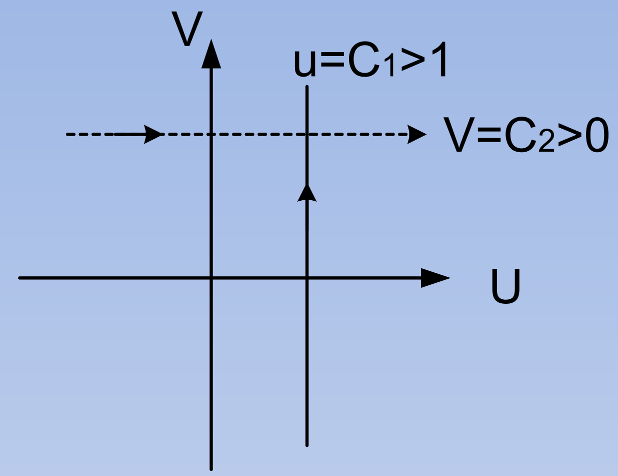
$$w = z^2$$

$$u = x^2 - y^2$$

$$v = 2xy$$

a hyperbola $x^2 - y^2 = c_1$ is mapped in a one to one manner onto the line $u = c_1$



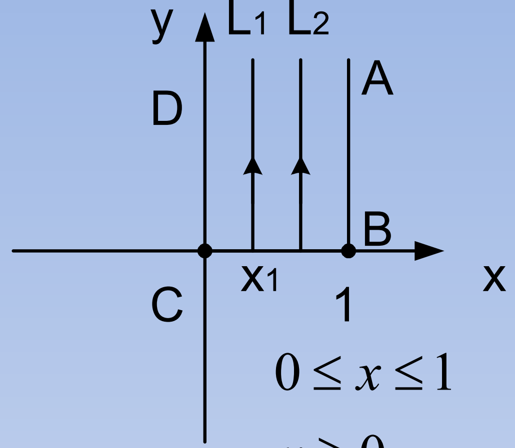


image

right hand branch x>0, u=C_1, $V = 2y\sqrt{y^2 + c_1}$ $(-\infty < y < \infty)$

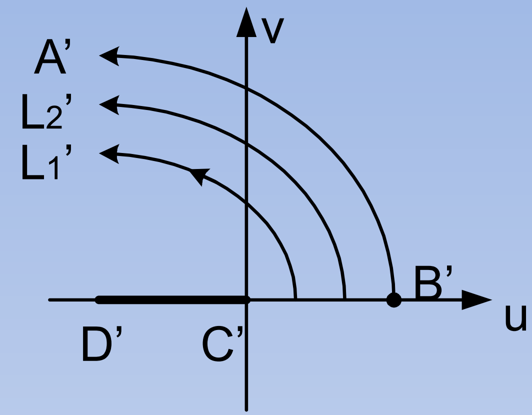
left hand branch x<0 u=C_1, $V = -2y\sqrt{y^2 + c_1}$ $(-\infty < y < \infty)$

Ex 2.



$0 \le x \le 1$

$y \ge 0$



When $0 < x_1 < 1$, point (x_1, y) moves up a vertical half line, L1, as y increases from y = 0.

$$u = x^2 - y^2, \qquad v = 2x_1 y$$

$$y = \frac{v}{2x_1}$$

$$u = x_1^2 - \left(\frac{v}{2x_1}\right)^2, \qquad v^2 = -4x_1^2\left(u - x_1^2\right)$$ $\longleftarrow$ a parabola with vertex at $\left(x_1^2, 0\right)$

half line CD is mapped of half line C'D'

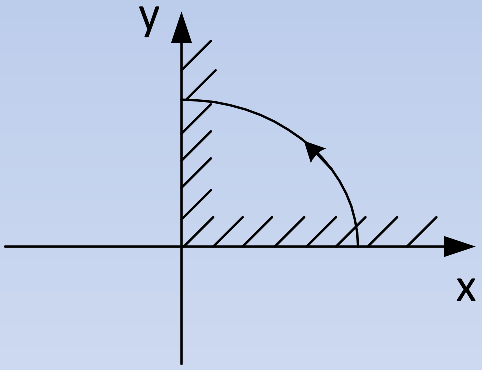
$(0, y)$ $\qquad\qquad$ $\left(-y^2, 0\right)$

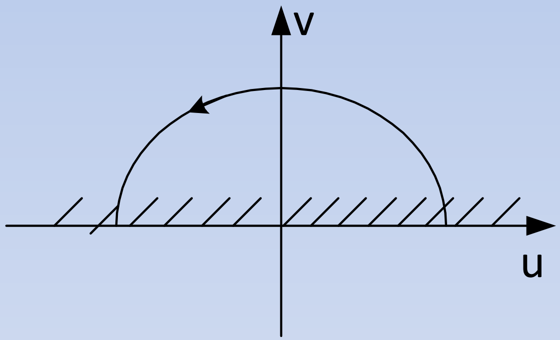
Ex 3.

$$w = z^2 = r^2 e^{i2\theta}$$

$$let \quad w = \rho e^{i\phi}$$

$$\rho = r^2, \qquad \phi = 2\theta + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$





$$r \geq 0, \quad 0 \leq \theta \leq \pi/2 \quad \xrightarrow{\text{one to one}} \quad \rho \geq 0, \quad 0 \leq \phi \leq \pi$$

Limits

Let a function f be defined at all points z in some deleted neighborhood of z_0

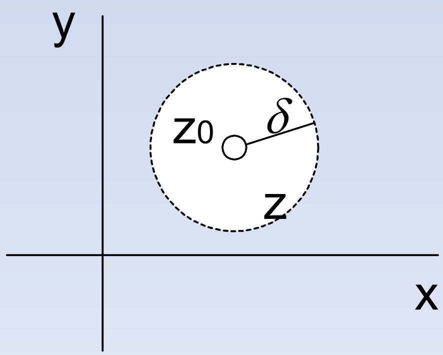
$$\lim_{z \to z_0} f(z) = w_0 \qquad (1)$$

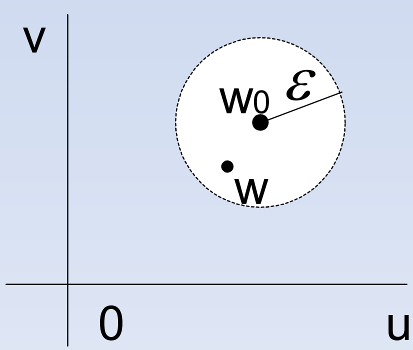
means: the limit of $f(z)$ as z approaches z_0 is w_0

$w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it.

(1) means that, for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad whenver \quad 0 < |z - z_0| < \delta \qquad (2)$$





Note:

(2) requires that *f* be defined at all points in some deleted neighborhood of z_0

such a deleted neighborhood always exists when z_0 is an interior point of a region on which f is defined. We can extend the definition of limit to the case in which z_0 is a boundary point of the region by agreeing that left of (2) be satisfied by only those points *z* that lie in both the region and the domain

$$0 < |z - z_0| < \delta$$

Example 1. show if

$$f(z) = \frac{iz}{2} \quad in \quad |z| < 1, \quad then$$

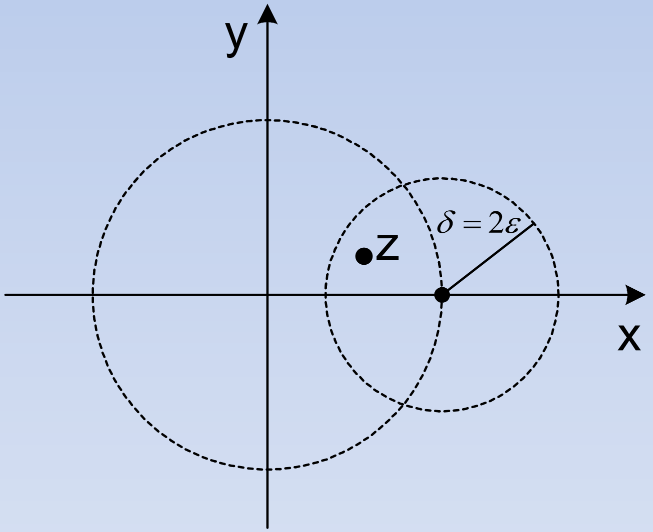
$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

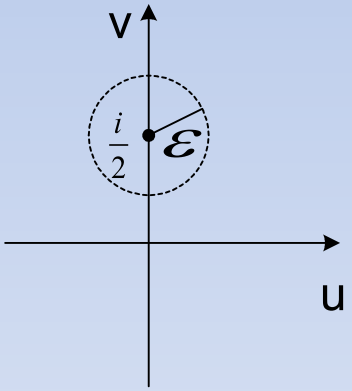
$$when \quad z \quad in \quad |z| < 1$$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}$$

For any such *z* and any positive number ε

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad 0 < |z-1| < 2\varepsilon$$





When a limit of a function $f(z)$ exists at a point z_0, it is unique.

If not, suppose $\lim_{z \to z_0} f(z) = w_0$, and $\lim_{z \to z_0} f(z) = w_1$

Then $\left| f(z) - w_0 \right| < \varepsilon \quad whenever \quad 0 < \left| z - z_0 \right| < \delta_0$

$$\left| f(z) - w_1 \right| < \varepsilon \quad whenever \quad 0 < \left| z - z_0 \right| < \delta_1$$

Let $\delta = \min\left(\delta_0, \delta_1\right)$

$$if \quad 0 < \left| z - z_0 \right| < \delta$$

$$\left| \left[f(z) - w_0 \right] - \left[f(z) - w_1 \right] \right| \le \left| f(z) - w_0 \right| + \left| f(z) - w_1 \right| < 2\varepsilon$$

But $\left| w_1 - w_0 \right| < 2\varepsilon$

Hence $\left| w_1 - w_0 \right|$ is a nonnegative constant, and ε can be chosen arbitrarily small.

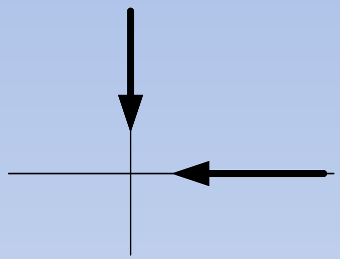
$$w_1 - w_0 = 0, \quad or \quad w_1 = w_0$$

Ex 2. If $f(z)=\frac{z}{\bar{z}}$ (4)

then $\lim_{z\to 0} f(z)$ does not exist.

show: $when \quad z=(x,0) \quad f(z)=\frac{x+i0}{x-i0}=1$

$when \quad z=(0,y) \quad f(z)=\frac{0+iy}{0-iy}=-1$



since a limit is unique, limit of (4) does not exist.

(2) provides a means of testing whether a given point w_0 is a limit, it does not directly provide a method for determining that limit.

Theorems on limits

Thm 1. Suppose that

$$f(z) = u(x,y) + iv(x,y), \qquad z_0 = x_0 + iy_0$$

$$\textit{and} \quad w_0 = u_0 + iv_0$$

Then $\lim\limits_{z \to z_0} f(z) = w_0$ iff

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \qquad \textit{and} \qquad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

pf : " $\Leftarrow$ "

$$|u - u_0| < \frac{\varepsilon}{2} \quad \textit{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1$$

$$|v - v_0| < \frac{\varepsilon}{2} \quad \textit{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2$$

$$\textit{let} \quad \delta = \min(\delta_1, \delta_2)$$

since

$$\left|(u+iv)-(u_0+iv_0)\right| = \left|(u-u_0)+i(v-v_0)\right| \le |u-u_0|+|v-v_0|$$

and

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = \left|(x-x_0)+i(y-y_0)\right| = \left|(x+iy)-(x_0+iy_0)\right|$$

$$\therefore \quad \left|(u+iv)-(u_0-iv_0)\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

$$whenever \quad 0 < \left|(x+iy)-(x_0+iy_0)\right| < \delta$$

“ ”

$\Rightarrow$

But $\left|(u+iv)-(u_0-iv_0)\right| < \varepsilon \quad whenever \quad 0 < \left|(x+iy)-(x_0+iy_0)\right| < \delta$

$$|u-u_0| \le \left|(u-u_0)+i(v-v_0)\right| = \left|(u+iv)-(u_0+iv_0)\right| < \varepsilon$$

and $|v-v_0| \le \left|(u-u_0)+i(v-v_0)\right| = \left|(u+iv)-(u_0+iv_0)\right| < \varepsilon$

$$\left|(x+iy)-(x_0+iy_0)\right| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$\therefore |u-u_0| < \varepsilon \quad and \quad |v-v_0| < \varepsilon$$

$$whenever \quad 0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

Thm 2. suppose that

$$\lim_{z \to z_0} f(z) = w_0 \qquad and \qquad \lim_{z \to z_0} F(z) = W_0 \qquad (7)$$

$$Then \quad \lim_{z \to z_0} \left[f(z) + F(z) \right] = w_0 + W_0$$

$$\lim_{z \to z_0} \left[f(z) \cdot F(z) \right] = w_0 W_0 \qquad (9)$$

$$and \quad if \quad W_0 \neq 0$$

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$$

pf: utilize Thm 1.

for (9).

$$f(z) = u(x, y) + iv(x, y)$$

$$F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0, \qquad w_0 = u_0 + iv_0, \qquad W_0 = U_0 + iV_0$$

use Thm 1. and (7)

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$ have the limits

$$\Downarrow \qquad \Downarrow$$

$$u_0U_0 - v_0V_0 \qquad v_0U_0 + u_0V_0$$
$$= w_0W_0$$

An immediate consequence of Thm. 1:

- $\lim\limits_{z \to z_0} c = c$
- $\lim\limits_{z \to z_0} z = z_0$
- $\lim\limits_{z \to z_0} z^n = z_0{}^n \qquad (n = 1, 2, \ldots)$ by property (9) and math induction.
- $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$

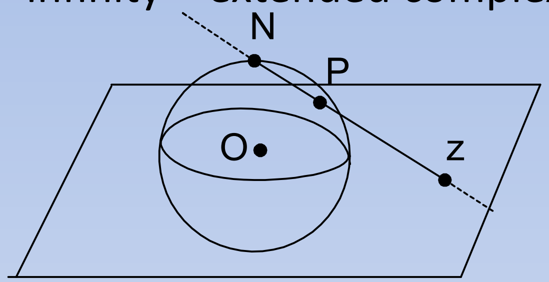
  $\lim\limits_{z \to z_0} P(z) = P(z_0)$ (11)
- $if \quad \lim\limits_{z \to z_0} f(z) = w_0, then \quad \lim\limits_{z \to z_0} |f(z)| = |w_0|$

  $\big||f(z)| - |w_0|\big| \le |f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

Limits involving the point at Infinity

It is sometime convenient to include with the complex plane the point at infinity, denoted by ∞, and to use limits involving it.

Complex plane + infinity = extended complex plane.



complex plane passing thru the equator of a unit sphere.

To each point *z* in the plane there corresponds exactly one point P on the surface of the sphere.

$\downarrow$

intersection of the line *z*-N with the surface.

$\uparrow$

north pole

To each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one point *z* in the plane.

By letting the point N of the sphere correspond to the point at infinity, we obtain a one-to-one correspondence between the points of the sphere and the points of the extended complex plane.

upper sphere $\longleftrightarrow$ exterior of unit circle

$|z| > \frac{1}{\varepsilon}$ $\longleftrightarrow$ points on the sphere close to N

$\downarrow$

ε neighborhood of ∞

- $\lim_{z \to z_0} f(z) = \infty$

$$\Leftrightarrow \quad |f(z)| > \frac{1}{\varepsilon} \quad \textit{whenever} \quad 0 < |z - z_0| < \delta$$

$$\Leftrightarrow \quad \left| \frac{1}{f(z)} - 0 \right| < \varepsilon \quad \textit{whenever} \quad 0 < |z - z_0| < \delta$$

$$\therefore \lim_{z \to z_0} f(z) = \infty \quad \textit{iff} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

$$Ex1. \quad \lim_{z \to -1} \frac{iz+3}{z+1} = \infty \quad \text{since} \quad \lim_{z \to -1} \frac{z+1}{iz+3} = 0$$

- $$\lim_{z\to\infty} f(z) = w_0$$

$$\Leftrightarrow \quad |f(z) - w_0| < \varepsilon \quad \textit{whenever} \quad |z| > \frac{1}{\delta}$$

$$\Leftrightarrow \quad \left|f\left(\frac{1}{z}\right) - w_0\right| < \varepsilon \quad \textit{whenever} \quad 0 < |z - 0| < \delta$$

$$\therefore \quad \lim_{z\to\infty} f(z) = w_0 \quad \textit{iff} \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

Ex 2.

$$\lim_{z\to\infty} \frac{2z+i}{z+1} = 2 \quad \text{since} \quad \lim_{z\to 0} \frac{\left(\frac{2}{z}\right)+i}{\left(\frac{1}{z}\right)+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = 2$$

- $$\lim_{z\to\infty} f(z) = \infty$$

$$\Leftrightarrow \quad |f(z)| > \frac{1}{\varepsilon} \quad \textit{whenever} \quad |z| > \frac{1}{\delta}$$

$$\Leftrightarrow \quad \left|f(\frac{1}{z})\right| > \frac{1}{\varepsilon} \quad \textit{whenever} \quad \left|\frac{1}{z}\right| > \frac{1}{\delta}$$

$$\Leftrightarrow \quad \left|\frac{1}{f(1/z)} - 0\right| < \varepsilon \quad \textit{whenever} \quad 0 < |z - 0| < \delta$$

$$\therefore \quad \lim_{z\to\infty} f(z) = \infty \quad \textit{iff} \quad \lim_{z\to 0} \frac{1}{f(\frac{1}{z})} = 0$$

Ex 3.

$$\lim_{z\to\infty} \frac{2z^3 - 1}{z^2 + 1} = \infty$$

$$\text{since} \quad \lim_{z\to 0} \frac{\frac{1}{z^2} + 1}{\frac{2}{z^3} - 1} = \lim_{z\to 0} \frac{z + z^3}{2 - z^3} = 0$$

Continuity

A function *f* is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) \quad \text{exists,} \qquad (1)$$

$$f(z_0) \qquad \text{exists,} \qquad (2)$$

$$\lim_{z \to z_0} f(z) = f(z_0) \qquad (3)$$

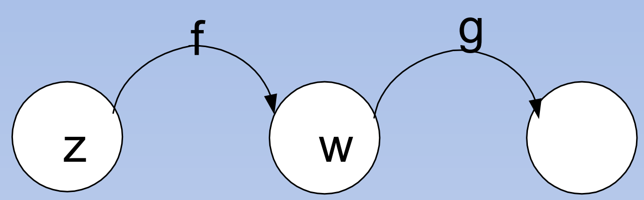
((3) implies (1)(2))

$$(\left|f(z) - f(z_0)\right| < \varepsilon \qquad whenever \quad \left|z - z_0\right| < \delta)$$

- if f_1, f_2 continuous at z_0, then $f_1 + f_2, f_1 \cdot f_2$ are also continuous at z_0.

So is $\dfrac{f_1}{f_2}$ $\quad if \quad f_2(z_0) \neq 0$

- A polynomial is continuous in the entire plane because of (11), section 12. p.37
- A composition of continuous function is continuous.



$$\left|g[f(z)] - g[f(z_0)]\right| < \varepsilon, \quad whenver \quad |f(z) - f(z_0)| < r,$$

$$whenver \quad |z - z_0| < \delta$$

- If a function *f(z)* is continuous and non zero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.

when $f(z_0) \neq 0$ let $\varepsilon = \dfrac{|f(z_0)|}{2}$

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad whenever \quad |z - z_0| < \delta$$

if there is a point *z* in the $|z - z_0| < \delta$ at which $f(z_0) = 0$ then

$|f(z_0)| < \dfrac{|f(z_0)|}{2}$, a contradiction.

From Thm 1

a function *f* of a complex variable is continuous at a point $z_0 = (x_0, y_0)$

iff its component functions *u* and *v* are continuous there.

Ex. The function

$$f(z) = \cos(x^2 - y^2)\cosh 2xy - i\sin(x^2 - y^2)\sinh 2xy$$

is continuous everywhere in the complex plane since

(i) $x^2 - y^2$ are continuous (polynomial)

$2xy$

(ii) cos, sin, cosh, sinh are continuous

(iii) real and imaginary component are continuous

$\longrightarrow$ complex function is continuous.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Derivatives

Let f be a function whose domain of definition contain a neighborhood of a point z_0. The <u>derivative</u> of f at z_0, written $f'(z_0)$, is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided this limit exists.

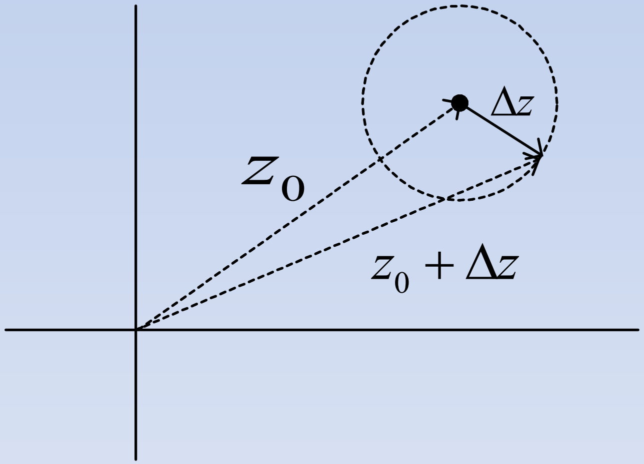
f is said to be differentiable at z_0.

$$let \quad \Delta z = z - z_0$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$let \quad \Delta w = f(z + \Delta z) - f(z).$$

$$f'(z) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$



Ex1. Suppose $f(z)=z^2$ at any point z

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z}=\lim_{\Delta z\to 0}(2z+\Delta z)=2z+0=2z$$

since $2z+\Delta z$ a polynomial in Δz.

$$\therefore\quad f'(z)=\frac{dw}{dz}=2z$$

Ex2. $f(z)=|z|^2$

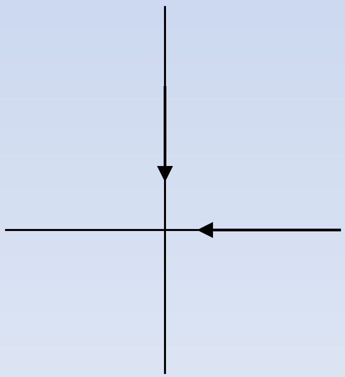
$$\frac{\Delta w}{\Delta z}=\frac{|z+\Delta z|^2-|z|^2}{\Delta z}=\frac{(z+\Delta z)\left(\bar{z}+\overline{\Delta z}\right)-z\bar{z}}{\Delta z}=\bar{z}+\overline{\Delta z}+z\frac{\overline{\Delta z}}{\Delta z}$$

when $\Delta z\to 0$ thru $(\Delta x,0)$ on the real axis $\overline{\Delta z}=\Delta z$

Hence if the limit of $\frac{\Delta w}{\Delta z}$ exists, its value = $\bar{z}+z$

when $\Delta z\to 0$ thru $(0,\Delta y)$ on the imaginary axis.

$\overline{\Delta z}=-\Delta z$, limit = $\bar{z}-z$ if it exists.



since limits are unique,

$\bar{z} + z = \bar{z} - z, \quad or \quad z = 0$ if $\frac{dw}{dz}$ is to exist.

observe that $\frac{\Delta w}{\Delta z} \to \overline{\Delta z}$ when $z = 0$

$\therefore \frac{dw}{dz}$ exists only at $z = 0$, its value = 0

- Example 2 shows that

  a function can be differentiable at a certain point but nowhere else in any neighborhood of that point.

- Re $|z|^2 = x^2 + y^2$ are continuous, partially

  Im $|z|^2 = 0$ differentiable at a point.

  but $|z|^2$ may not be differentiable there.

- $f(z)=|z|^2$ is continuous at each point in the plane since its components are continuous at each point.

$\therefore$ continuity $\overset{\text{not necessarily}}{\not\Longrightarrow}$ derivative exists.

existence of derivative $\Longrightarrow$ continuity.

$$\lim_{z\to z_0}\left[f(z)-f(z_0)\right]=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\lim_{z\to z_0}(z-z_0)$$

$$=f'(z_0)\cdot 0=0$$

$$\therefore \lim_{z\to z_0}f(z)=f(z_0)$$

16. Differentiation Formulas

$$\frac{d}{dz}C = 0 \qquad C: \text{complex} \quad \text{constant}$$

$$\frac{d}{dz}z = 1$$

$$\frac{d}{dz}\left[cf(z)\right] = cf'(z)$$

$$\frac{d}{dz}z^n = nz^{n-1}$$ *n* a positive integer.

$$\frac{d}{dz}\left[f(z) + F(z)\right] = f'(z) + F'(z)$$

$$\frac{d}{dz}\left[f(z)F(z)\right] = f(z)F'(z) + f'(z)F(z) \qquad (4)$$

$$when \quad F(z) \neq 0$$

$$\frac{d}{dz}\left[\frac{f(z)}{F(z)}\right] = \frac{F(z)f'(z) - f(z)F'(z)}{\left[F(z)\right]^2}$$

$pf:(4)$

$$f(z+\Delta z)F(z+\Delta z)-f(z)F(z)$$
$$=f(z)[F(z+\Delta z)-F(z)]+[f(z+\Delta z)-f(z)]F(z+\Delta z)$$
$$\frac{f(z+\Delta z)F(z+\Delta z)-f(z)F(z)}{\Delta z}=f(z)\frac{F(z+\Delta z)-F(z)}{\Delta z}+\frac{f(z+\Delta z)-f(z)}{\Delta z}F(z+\Delta z)$$

$$as \quad \Delta z\to 0 \qquad \frac{d}{dz}[fF]=f(z)F'(z)+f'(z)F(z+\Delta z)$$
$$=f(z)F'(z)+f'(z)F(z)$$ (F continuous at z)

f has a derivative at z_0

g has a derivative at $f(z_0)$

$F(z)=g[f(z)]$ has a derivative at z_0

and $$F'(z_0)=g'[f(z_0)]f'(z_0)$$ chain rule (6)

$$\frac{dW}{dz}=\frac{dW}{dw}\frac{dw}{dz}$$

pf of (6)

choose a z_0 at which $f'(z_0)$ exists.

let $w_0 = f(z_0)$ and assume $g'(w_0)$ exists.

Then, there is $|w-w_0|<\varepsilon$ of w_0 such that

we can define a function Φ, with $\Phi(w_0)=0$ and

$$\Phi(w)=\frac{g(w)-g(w_0)}{w-w_0}-g'(w_0) \qquad when \quad w\neq w_0 \qquad (7)$$

$\lim_{w\to w_0}\Phi(w)=0,$ Hence Φ is continuous at w_0

$$(7) \Rightarrow \quad g(w) - g(w_0) = \left[g'(w_0) + \Phi(w)\right](w - w_0) \quad (|w - w_0| < \varepsilon) \quad (9)$$

valid even when $w = w_0$

since $f'(z_0)$ exists and therefore f is continuous at z_0, then we can have $f(z)$ lies in $|w - w_0| < \varepsilon$ $\quad of \quad w_0 \quad if \quad |z - z_0| < \delta$

substitute w by $f(z)$ in (9) when z in $|z - z_0| < \delta$

(9) becomes

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\} \frac{f(z) - f(z_0)}{z - z_0} \quad (10)$$

$$(0 < |z - z_0| < \delta)$$

since f is continuous at z_0, Φ is continuous at $w_0 = f(z_0)$

$\therefore \quad \Phi[f(z)]$ is continuous at z_0, and since $\Phi(w_0) = 0$

$$\lim_{z \to z_0} \Phi\left[f(z)\right] = 0$$

so (10) becomes $F'(z_0) = g'\left[f(z_0)\right] f'(z_0)$ as $z \to z_0$

Cauchy-Riemann Equations

Suppose that $f'(z_0) = \lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ *exists.*

writing $z_0 = x_0 + iy_{0,}$ $\quad \Delta z = \Delta x + i\Delta y$

Then by Thm. 1

$$\mathrm{Re}\left[f'(z_0)\right] = \lim_{(\Delta x, \Delta y) \to (0,0)} \mathrm{Re}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \qquad (3)$$

$$\mathrm{Im}\left[f'(z_0)\right] = \lim_{(\Delta x, \Delta y) \to (0,0)} \mathrm{Im}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \qquad (4)$$

where

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} \qquad (5)$$

$$+ \frac{i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}$$

Let $(\Delta x, \Delta y)$ tend to (0,0) horizontally through $(\Delta x, 0)$ i.e. $\Delta y = 0$

$$\therefore \quad \mathrm{Re}[f'(z_0)] = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$\mathrm{Im}[f'(z_0)] = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$\therefore f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0) \qquad (6)$$

Let $(\Delta x, \Delta y)$ tend to (0,0) vertically thru $(0, \Delta y)$ i.e. $\Delta x = 0$ then

$$f'(z_0) = \left(\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + \frac{i\left[v(x_0, y_0 + \Delta y) - v(x_0, y_0)\right]}{i\Delta y}\right)$$

$$= v_y(x_0, y_0) - i u_y(x_0, y_0) \qquad (7)$$

$$= -i u_y + v_y$$

(6)= (7)

$$\therefore \quad u_x(x_0, y_0) = v_y(x_0, y_0) \qquad (8)$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

Cauchy-Riemann Equations.

Thm : suppose $f(z) = u(x,y) + iv(x,y)$

$f'(z)$ exists at a point $z_0 = x_0 + iy_0$

Then u_x, u_y, v_x, v_y exist at (x_0, y_0)

and $u_x = v_y,\ u_y = -v_x;$ also $f'(z) = u_x + iv_x$

Ex 1. $f(z) = z^2 = x^2 - y^2 + i2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

$u_x = v_y, \qquad u_y = -v_x$

$f'(z) = 2x + i2y = 2(x+iy) = 2z$

Cauchy-Riemann equations are Necessary conditions for the existence of the derivative of a function f at z_0.

⇨ Can be used to locate points at which f does not have a derivative.

Ex 2. $f(z) = |z|^2$,

$u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$

$u_x = 2x \qquad v_x = 0 \qquad\qquad u_x \neq v_y, \qquad f'(z)$ does not exist

$u_y = 2y \qquad v_y = 0$ at any nonzero point.

The above Thm does not ensure the existence of $f'(z_0)$
(say)

Sufficient Conditions For Differentiability

$$f'(z_0) \quad \text{exist} \quad \rightarrow u_x = v_y, \quad u_y = -v_x$$

but not

$$" \leftarrow "$$

Thm.

Let $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$

suppose u_x, u_y, v_x, v_y exist everywhere in the neighborhood and are continuous at (x_0, y_0)

Then, if $u_x = v_y, \quad u_y = -v_x \quad \text{at } (x_0, y_0)$

$$\Rightarrow f'(z_0) \quad exists.$$

$$pf : let \qquad \Delta z = \Delta x + i\Delta y, \qquad where \quad 0 < |\Delta z| < \varepsilon$$

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

Thus $\Delta w = \Delta u + i\Delta v \quad \Leftarrow u(z_0 + \Delta z) - u(z_0) + i[v(z_0 + \Delta z) - v(z_0)]$

where

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$$

Now in view of the continuity of the first-order partial derivatives of *u* and v at the point (x_0, y_0)

$$\Delta u = u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + u_{xy}(x_0, y_0)\Delta x\Delta y + u_{xx}(x_0, y_0)\frac{\Delta x^2}{2!} + u_{yy}(x_0, y_0)\frac{\Delta y^2}{2!} + \ldots - u(x_0, y_0)$$

$$= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_2\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\varepsilon_1, \varepsilon_2 \to 0, as \quad (\Delta x, \Delta y) \to (0,0)$$

$$\Delta w = \Delta u + i\Delta v$$

$\longleftarrow$ where ε_1 and ε_2 tend to 0 as $(\Delta x, \Delta y) \to (0,0)$ in the Δz-plane.

$$= \quad above \qquad (3)$$

assuming that the Cauchy-Riemann equations are satisfied at (x_0, y_0) we can replace

$u_y \quad by \quad -v_x, \quad and \quad v_y \quad by \quad u_x$ in (3), and divide thru by Δz

to get
$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_2)\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z} \qquad (4)$$

$$but \quad \sqrt{(\Delta x)^2 + (\Delta y)^2} = |\Delta z|$$

$$so \quad \left|\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z}\right| = 1$$

also $\varepsilon_1 + i\varepsilon_2$ tends to 0, as $(\Delta x, \Delta y) \to (0,0)$

The last term in(4) tends to 0 as $\Delta z \to 0$

$$\therefore \text{ The limit of } \frac{\Delta w}{\Delta z} \text{ exists, and } f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Ex 1. $f(z) = e^x(\cos y + i \sin y)$

$$u(x,y) = e^x \cos y$$

$$v(x,y) = e^x \sin y$$

$u_x = v_y, \qquad u_y = -v_x$ everywhere, and continuous.

$\Rightarrow f'(z)$ exists everywhere, and

$$f'(z) = u_x + iv_x = e^x(\cos y + i \sin y)$$

Ex 2. $f(z) = |z|^2$

$u(x,y) = x^2 + y^2 \qquad u_x = 2x \qquad u_y = 2y$

$v(x,y) = 0 \qquad v_x = 0 \qquad v_y = 0$

has a derivative at z=0.

$$f'(0) = 0 + i0$$

can not have derivative at any nonzero point.

Polar Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = x + iy = re^{i\theta} \qquad (z \neq 0)$$

Suppose that u_x, u_y, v_x, v_y exist everywhere in some neighborhood of a given non-zero point z_0 and are continuous at that point.

$u_r, u_\theta, v_r, v_\theta$ also have these properties, and (by chain rule)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$u_r = u_x \cos\theta + u_y \sin\theta \qquad (2)$$

$$u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

Similarly,

$$v_r = v_x \cos\theta + v_y \sin\theta \qquad (3)$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta$$

If $\quad u_x = v_y,\ u_y = -v_x$

$$v_r = -u_y \cos\theta + u_x \sin\theta \qquad (5)$$

$$v_\theta = u_y r \sin\theta + u_x r \cos\theta$$

from (2) (5), $\quad u_r = \frac{1}{r} v_\theta \qquad \text{at } z_0 \qquad (6)$

$$\frac{1}{r} u_\theta = -v_r$$

Thm. p53…

$$f'(z_0) = u_x + iv_x$$
$$= ?$$

$$u_r = u_x \cos\theta + u_y \sin\theta$$
$$u_r \cos\theta = u_x \cos^2\theta + u_y \sin\theta \cos\theta$$

$$v_r = v_x \cos\theta + v_y \sin\theta$$
$$= -u_y \cos\theta + u_x \sin\theta$$
$$v_r \sin\theta = -u_y \cos\theta \sin\theta + u_x \sin^2\theta$$

$$\boxed{\therefore u_r \cos\theta + v_r \sin\theta = u_x}$$

$$u_r = v_y \cos\theta - v_x \sin\theta$$
$$u_r \sin\theta = v_y \cos\theta \sin\theta - v_x \sin^2\theta$$

$$v_r = v_x \cos\theta + v_y \sin\theta$$
$$\cos\theta v_r = v_x \cos^2\theta + v_y \sin\theta \cos\theta$$

$$\boxed{v_r \cos\theta - u_r \sin\theta = v_x}$$

$$\therefore f'(z_0) = u_r \cos\theta + v_r \sin\theta + i\left(v_r \cos\theta - u_r \sin\theta\right)$$
$$= (\cos\theta - i\sin\theta)(u_r + iv_r)$$
$$= e^{-i\theta}(u_r + iv_r) \qquad (7)$$

Ex : Consider $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}}$

$$u(r,\theta) = \frac{1}{r}\cos\theta \qquad v(r,\theta) = -\frac{1}{r}\sin\theta$$

$$u_r = -\frac{1}{r^2}\cos\theta \qquad v_r = \frac{1}{r^2}\sin\theta$$

$$u_\theta = -\frac{1}{r}\sin\theta \qquad v_\theta = -\frac{1}{r}\cos\theta$$

$\Rightarrow u_r = \frac{1}{r}v_\theta, \quad \frac{1}{r}u_\theta = -v_r$ at any non-zero point $z = re^{i\theta}$

$\therefore f'$ exists

$$f' = e^{-i\theta}\left(-\frac{1}{r^2}\cos\theta + \frac{i}{r^2}\sin\theta\right)$$

$$= \frac{1}{r^2}(-e^{-i\theta})e^{-i\theta} = -\frac{1}{r^2}e^{-i2\theta} = -\frac{1}{z^2}$$