

# **DERIVATIVE OF ANALYTIC FUNCTION**

# Derivatives of Analytic Functions

$$z = x + iy$$

Let  $f(z)$  be analytic around  $z$ , then

$$\frac{df(x)}{dx} = g(x) \quad \rightarrow \quad \frac{df(z)}{dz} = g(z)$$

**Proof :**

$$f(z) \text{ analytic} \quad \rightarrow \quad f'(z) = \frac{\partial f(x+iy)}{\partial x} = \left. \frac{df(x)}{dx} \right|_{x=z} = g(z)$$

$$\text{E.g.} \quad \frac{dx^n}{dx} = n x^{n-1} \quad \rightarrow \quad \frac{dz^n}{dz} = n z^{n-1}$$

$\therefore$  Analytic functions can be defined by Taylor series of the same coefficients as their real counterparts.

# Derivative of Logarithm

CRCs  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$   $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$r = \sqrt{x^2 + y^2}$   $\theta = \tan^{-1} \frac{y}{x}$

$$\frac{d \ln z}{dz} = \frac{1}{z}$$

for  $z$  within each branch.

**Proof :**  $\ln z = \ln r + i(\theta + 2\pi n) = u + iv \rightarrow$

$u = \ln r$   
 $v = \theta + 2\pi n$

$$\frac{\partial u}{\partial x} = \frac{1}{r} \frac{\partial r}{\partial x} = \frac{x}{r^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial \theta}{\partial y} = \left(1 + \frac{y^2}{x^2}\right)^{-1} \frac{1}{x} = \frac{x}{r^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{r} \frac{\partial r}{\partial y} = \frac{y}{r^2} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial \theta}{\partial x} = \left(1 + \frac{y^2}{x^2}\right)^{-1} \left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$$

$\rightarrow$   $\ln z$  is analytic within each branch.

$$\therefore \frac{d \ln z}{dz} = \frac{\partial \ln z}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{x}{r^2} - i \frac{y}{r^2} = \frac{1}{x + iy} = \frac{1}{z}$$

$r^2 = z z^*$

QED

# Derivatives of Functions $w(t)$

- Consider derivatives of complex-valued functions  $w$  of real variable  $t$

$$w(t) = u(t) + iv(t)$$

where the function  $u$  and  $v$  are real-valued functions of  $t$ .  
The derivative

$$w'(t), \text{ or } \frac{d}{dt} w(t)$$

of the function  $w(t)$  at a point  $t$  is defined as

$$w'(t) = u'(t) + iv'(t)$$

# Derivatives of Functions $w(t)$

- Properties

For any complex constant  $z_0 = x_0 + iy_0$ ,

$$\begin{aligned}\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u + iv)]' = [\overset{u(t)}{\underbrace{(x_0 u - y_0 v)}} + i \overset{v(t)}{\underbrace{(y_0 u + x_0 v)}}]' \\ &= (x_0 u - y_0 v)' + i(y_0 u + x_0 v)' \\ &= (x_0 u' - y_0 v') + i(y_0 u' + x_0 v') \\ &= (x_0 + iy_0)(u' + iv') = z_0 w'(t)\end{aligned}$$



# Derivatives of Functions w(t)

- Properties

$$\frac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}$$

where  $z_0 = x_0 + iy_0$ . We write

$$e^{z_0 t} = e^{(x_0 + iy_0)t} = \overset{u(t)}{e^{x_0 t} \cos y_0 t} + i \overset{v(t)}{e^{x_0 t} \sin y_0 t}$$

$$\frac{d}{dt} e^{z_0 t} = (e^{x_0 t} \cos y_0 t)' + i(e^{x_0 t} \sin y_0 t)'$$

Similar rules from calculus and some simple algebra then lead us to the expression

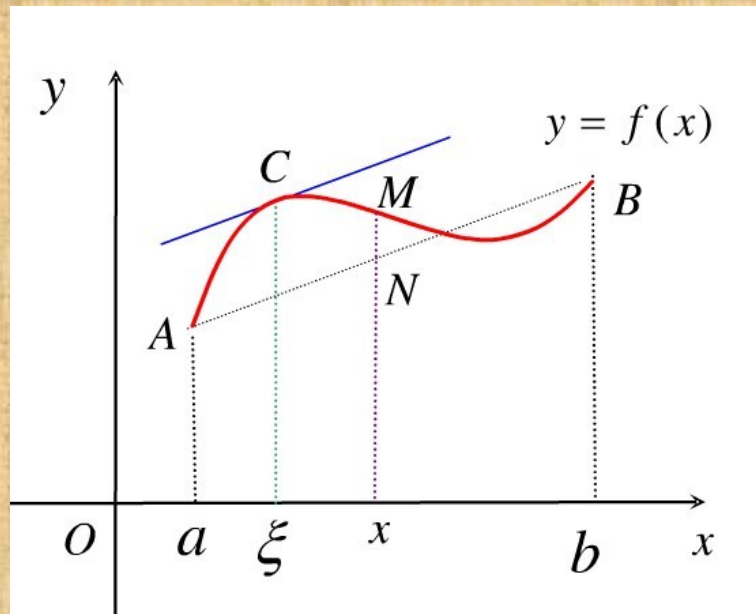
$$\frac{d}{dt} e^{z_0 t} = (x_0 + iy_0) e^{(x_0 + iy_0)t} = z_0 e^{z_0 t}$$

# Derivatives of Functions w(t)

- Example

Suppose that the *real function*  $f(t)$  is continuous on an interval  $a \leq t \leq b$ , if  $f'(t)$  exists when  $a < t < b$ , the mean value theorem for derivatives tells us that there is a number  $\zeta$  in the interval  $a < \zeta < b$  such that

$$f'(\zeta) = \frac{f(b) - f(a)}{b - a}$$



# Derivatives of Functions $w(t)$

- Example (Cont')

The mean value theorem no longer applies for some *complex functions*. For instance, the function

$$w(t) = e^{it}$$

on the interval  $0 \leq t \leq 2\pi$ .

Please note that

$$|w'(t)| = |ie^{it}| = 1$$

And this means that the derivative  $w'(t)$  is never zero, while

$$w(2\pi) - w(0) = 0 \quad \Rightarrow \quad w'(\zeta) \neq \frac{w(2\pi) - w(0)}{2\pi - 0} = 0, \forall \zeta \in (0, 2\pi)$$

**Note: not every rules from calculus holds for complex functions**