

*Taylor's and Laurent's Series

*Taylor Series

$$f(z) = \frac{1}{2\pi i} \oint_C f(z') dz'$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

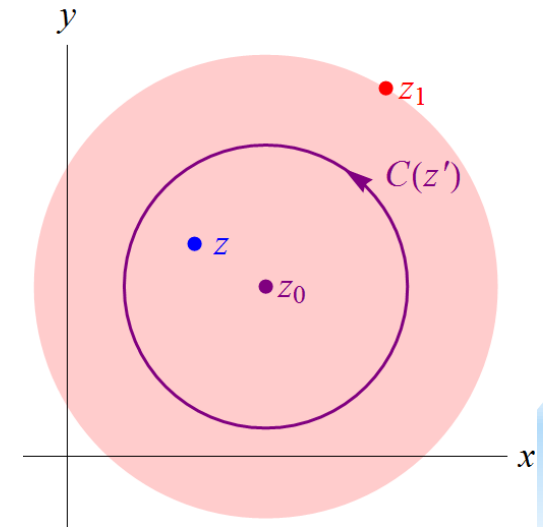
$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z_0} dz'$$

$$\frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{1}{z' - z_0} \left(1 - \frac{z - z_0}{z' - z_0} \right)^{-1} = \frac{1}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n$$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

$$\rightarrow f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) \quad \text{Taylor series}$$

(f analytic in $R \supset C$)



Let z_1 be the closest singularity from z_0 , then the **radius of convergence** is $|z_1 - z_0|$.

Engineering Mathematics III
i.e., series converges for $|z - z_0| < |z_1 - z_0|$

*Laurent Series

Let f be analytic within an annular region

$$r \leq |z - z_0| \leq R$$

$$\rightarrow f(z) = \frac{1}{2\pi i} \left[\oint_{C_1} f(z') dz' + \oint_{C_2} f(z') dz' \right]$$

$$C_1: \frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{1}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n$$

$$C_2: \frac{1}{z' - z} = -\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z' - z_0}{z - z_0} \right)^n$$

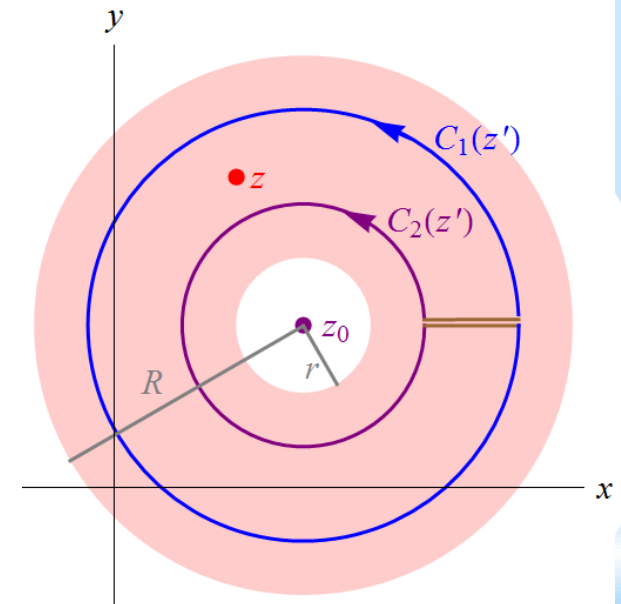
$\rightarrow$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} f(z') dz'$$

$$\frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} f(z') dz'$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C f(z') dz'$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C f(z') dz'$$



$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta + \frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$\sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

→

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Laurent series

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

C within f 's region of analyticity

*Laurent Expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C f(z) dz$$

$$f(z) = \frac{1}{z(z-1)}$$

Consider expansion about $z_0 = 0 \rightarrow f$ is analytic for $0 < |z| < 1$

Expansion via binomial theorem :

$$f(z) = -\left(\frac{1}{z} + \frac{1}{1-z}\right) = -\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n\right)$$

Laurent series :

$$a_n = \frac{1}{2\pi i} \oint_C f(z) dz = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_C z^k dz = \begin{cases} -1 & n \geq -1 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$

$$f(z) = -\sum_{n=-1}^{\infty} z^n$$