Complex Fourier Transforms

Complex Fourier transform

- Consider the expression

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} F_n \cos(n\omega_0 t) + iF_n \sin(n\omega_0 t)$$

$$= \sum_{n=0}^{\infty} (F_n + F_{-n}) \cos(n\omega_0 t) + i(F_n - F_{-n}) \sin(n\omega_0 t)$$

- So $a_n = F_n + F_{-n}$ and $b_n = i(F_n - F_{-n})$
- Since a_n and b_n are real, we can let $F_{-n} = \overline{F_n}$

and get $a_n = 2\,\mathrm{Re}(F_n)$ and $b_n = -2\,\mathrm{Im}(F_n)$

$$\mathrm{Re}(F_n) = \frac{a_n}{2} \quad \text{and} \quad \mathrm{Im}(F_n) = -\frac{b_n}{2}$$

- Thus $$F_n = \frac{1}{T}\left(\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\,dt - i\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\,dt \right)$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)(\cos(n\omega_0 t)\,dt - i\sin(n\omega_0 t))dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,e^{-in\omega_0 t}\,dt$$

$$= \left|F_n\right| e^{i\varphi_n}$$

- So you could also write $f(t) = \sum_{n=-\infty}^{\infty} \left|F_n\right| e^{i(n\omega_0 t + \varphi_n)}$

The Fourier transform G(k) and the original function g(x) are both in general complex.

$$\Im\{g(x)\} = G_r(k) + iG_i(k)$$

The Fourier transform can be written as,

$$\Im\{g(x)\} = G(k) = A(k)e^{i\Theta(k)}$$

$$A = |G| = \sqrt{G_r^2 + G_i^2}$$

$$A \equiv \text{ amplitude spectrum, or magnitude spectrum}$$

$$\Theta \equiv \text{ phase spectrum}$$

$$\mathrm{A}^2 = |G|^2 = G_r^2 + G_i^2 \equiv \text{ power spectrum}$$