Convolution Theorem

Convolution theorem

Theorem

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g)$$

$$\mathcal{F}^{-1}(F * G) = \mathcal{F}^{-1}(F)\mathcal{F}^{-1}(G)$$

$$\mathcal{F}^{-1}(FG) = \mathcal{F}^{-1}(F) * \mathcal{F}^{-1}(G)$$

Proof (1)

$$\mathcal{F}(f * g) = \int_{-\infty}^{\infty} f(t')g(t-t')\int_{-\infty}^{\infty} e^{-i\omega t}\,dt\,dt'$$

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt'\int_{-\infty}^{\infty} g(t-t')e^{-i\omega(t-t')}dt$$

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt'\int_{-\infty}^{\infty} g(t'')e^{-i\omega t''}dt''$$

$$= \mathcal{F}(f)\mathcal{F}(g)$$

Convolution Theorem
Example

$$f(x)=x \quad g(x)=e^{-x^2}$$

$$f*g=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\xi)e^{-\xi^2}d\xi$$

$$\int_{-\infty}^{\infty}\xi\, e^{-\xi^2}d\xi=0 \text{ by symmetry}$$

$$\int_{-\infty}^{\infty}e^{-\xi^2}d\xi=\sqrt{\pi}$$

$$\text{Hence } \; f*g=\frac{x\sqrt{\pi}}{\sqrt{2\pi}}=\frac{x}{\sqrt{2}}$$