

Introduction



Jean Baptiste Joseph Fourier
(Mar 21st 1768 – May 16th 1830)
French mathematician, physicist

Main Work:

Théorie analytique de la chaleur
(The Analytic Theory of Heat)

- Any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable (Incorrect)
- The concept of dimensional homogeneity in equations
- Proposal of his partial differential equation for conductive diffusion of heat

Discovery of the "greenhouse effect"

Fourier Transform

Transition from Fourier integral to Fourier transform

$$f(x) = \int_0^{\infty} [a(\omega)\cos\omega x + b(\omega)\sin\omega x]d\omega. \quad (1a)$$

$$\text{Where } a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos\omega x \, dx, \quad (1b)$$

$$b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin\omega x \, dx.$$

Put (1b) into (1a):

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} f(\xi) [\cos\omega\xi \cos\omega x + \sin\omega\xi \sin\omega x] d\xi \right\} d\omega \\ &= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(\xi) \cos\omega(\xi - x) d\xi d\omega \end{aligned}$$

Since $\cos(A-B)=\cos A \cos B + \sin A \sin B$ and introduce complex exponentials:

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(\xi) \frac{e^{i\omega(\xi-x)} + e^{-i\omega(\xi-x)}}{2} d\xi d\omega \\ &= \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{i\omega(\xi-x)} d\xi d\omega + \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} d\xi d\omega \end{aligned}$$

To combine the two terms on the right-hand side, let us change the dummy integration variable from ω to $-\omega$:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_0^{-\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} d\xi (-d\omega) \\ &\quad + \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} d\xi d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi \right] e^{i\omega x} d\omega \end{aligned}$$

Thus,

$$f(x) = a \int_{-\infty}^{\infty} c(\omega) e^{i\omega x} d\omega \quad c(\omega) = b \int_{-\infty}^{\infty} f(\xi) e^{-i\omega \xi} d\xi \quad ab = 1/2\pi$$

There is no longer a need to distinguish x and ξ , so:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega) e^{i\omega x} d\omega$$

$$c(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega) e^{i\omega x} d\omega, \quad (1)$$

$$c(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx. \quad (2)$$

They can be a transform pair: (2) defines the **Fourier transform**, $c(\omega)$, of the given function $f(x)$, and (1) is called the **inversion formula**.

$$F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx,$$

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega.$$

Example:

Derive the result $F\left\{e^{-a|x|}\right\} = \frac{2a}{\omega^2 + a^2} \quad (a>0)$

Solution:

According to the definition $F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx$

Then

$$F\left\{e^{-a|x|}\right\} = \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx = \int_{-\infty}^0 e^{ax} e^{-i\omega x} dx + \int_0^{\infty} e^{-ax} e^{-i\omega x} dx = \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2 + \omega^2}$$

Properties and applications

1. Linearity of the transform and its inverse.

$$F\{af+bg\}=aF\{f\}+bF\{g\}$$

$$F^{-1}\{a\hat{f}+b\hat{g}\}=aF^{-1}\{\hat{f}\}+bF^{-1}\{\hat{g}\}$$

2. Transform of n th derivative.

$$F\{f^{(n)}(x)\} = (i\omega)^n \hat{f}(\omega).$$

3. Fourier convolution.

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - \xi)g(\xi)d\xi.$$

Then the Fourier convolution theorem:

$$F\{f * g\} = \hat{f}(\omega)\hat{g}(\omega) \text{ and } F^{-1}\{\hat{f}\hat{g}\}=f * g$$

4. Translation formulas, x-shift and ω -shift

$$\begin{aligned} F\{f(x - a)\} &= e^{-ia\omega} \hat{f}(\omega) \\ F^{-1}\{\hat{f}(\omega - a)\} &= e^{ia\omega} f(x) \end{aligned}$$

Example:

Solve the wave equation $c^2 u_{xx} = u_{tt}$; $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$

Take the Fourier Transform of both equations. The initial condition gives

$$\hat{u}(\omega, 0) = \hat{f}(\omega)$$

$$\hat{u}_t(\omega, 0) = \left. \frac{\partial}{\partial t} \hat{u}(x, t) \right|_{t=0} = \hat{g}(\omega)$$

And the PDE gives

$$c^2 (-\omega^2 \hat{u}(\omega, t)) = \frac{\partial^2}{\partial t^2} \hat{u}(\omega, t)$$

Which is basically an ODE in t , we can write it as

$$\frac{\partial^2}{\partial t^2} \hat{u}(\omega, t) + c^2 \omega^2 \hat{u}(\omega, t) = 0$$

Which has the solution, and derivative

$$\hat{u}(\omega, t) = A(\omega) \cos c\omega t + B(\omega) \sin c\omega t$$

$$\frac{\partial}{\partial t} \hat{u}(\omega, t) = -c\omega A(\omega) \sin c\omega t + c\omega B(\omega) \cos c\omega t$$

So the first initial condition gives $A(\omega) = \hat{f}(\omega)$ and the second gives $c\omega B(\omega) = \hat{g}(\omega)$ and make the solution

$$\hat{u}(\omega, t) = \hat{f}(\omega) \cos c\omega t + \frac{\hat{g}(\omega)}{\omega} \frac{\sin c\omega t}{c}$$

Let's first look at

$$\hat{f}(\omega) \cos c\omega t = \hat{f}(\omega) \left(\frac{e^{i\omega ct} + e^{-i\omega ct}}{2} \right) = \frac{1}{2} (\hat{f}(\omega) e^{i\omega ct} + \hat{f}(\omega) e^{-i\omega ct})$$

Then

$$F^{-1}[\hat{f}(\omega) \cos c\omega t] = \frac{1}{2} (f(x+ct) + f(x-ct))$$

The second piece

$$\frac{\hat{g}(\omega)}{\omega} \frac{\sin c\omega t}{c} = \frac{\hat{g}(\omega)}{i\omega} \frac{\sin c\omega t}{-ic}$$

And now the first factor looks like an integral, as a derivative with respect to x would cancel the $i\omega$ in bottom. Define:

$$h(x) = \int_{s=0}^x g(s) ds$$

By fundamental theorem of calculus

$$h'(x) = g(x) \quad \hat{g}(\omega) = i\omega \hat{h}(\omega)$$

So

$$\frac{\hat{g}(\omega)}{\omega} \frac{\sin c\omega t}{c} = \hat{h}(\omega) \left(\frac{e^{ic\omega t} - e^{-ic\omega t}}{2i} \right) \frac{1}{-ic} = \frac{1}{2c} (\hat{h}(\omega) e^{ic\omega t} - \hat{h}(\omega) e^{-ic\omega t})$$

$$F^{-1} \left[\frac{1}{\omega c} \hat{g}(\omega) \sin c\omega t \right] = \frac{1}{2c} (h(x+ct) - h(x-ct)) = \frac{1}{2c} \left(\int_0^{x+ct} g(s) ds - \int_0^{x-ct} g(s) ds \right) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

Putting both piece together we get the solution

$$u(x, t) = \frac{1}{2} (f(x-ct) + f(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

Cosine and sine transforms

1. Fourier cosine transform

$$F_C\{f(x)\} = \hat{f}_C(\omega) = \int_0^{\infty} f(x) \cos \omega x dx,$$

And its inverse:

$$F_C^{-1}\{\hat{f}_C(\omega)\} = f(x) = \frac{2}{\pi} \int_0^{\infty} \hat{f}_C(\omega) \cos \omega x d\omega.$$

2. Fourier sine transform

$$F_S\{f(x)\} = \hat{f}_S(\omega) = \int_0^{\infty} f(x) \sin \omega x dx,$$

And its inverse:

$$F_S^{-1}\{\hat{f}_S(\omega)\} = f(x) = \frac{2}{\pi} \int_0^{\infty} \hat{f}_S(\omega) \sin \omega x d\omega$$

Properties:

$$F_C\{f'(x)\} = \omega \hat{f}_S(\omega) - f(0)$$

$$F_S\{f'(x)\} = -\omega \hat{f}_C(\omega).$$

$$F_C\{f''(x)\} = -\omega^2 \hat{f}_C(\omega) - f'(0).$$

$$F_S\{f''(x)\} = -\omega^2 \hat{f}_S(\omega) + \omega f(0)$$

Example:

Solve heat transfer equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ B.C: (1) $u(0,t)=0$
(2) $u(x,0)=P(x)$ $1 \leq x \leq 2$ or $P(x)=1$,

Solution with Fourier Sine Transform:

$$F_s\{u'' - u'\} = F_s\{0\}$$

$$F_s\{u''\} = F_s\{u'\}$$

$$-\omega^2 \hat{f}_s + \omega f_0 = -\omega \hat{f}_c(\omega) = \frac{\partial \hat{f}_s}{\partial t}$$

According to the B.C, we can get

$$f_0 = 0 \quad -\omega^2 \hat{f}_s(\omega, t) = \frac{\partial \hat{f}_s(\omega, t)}{\partial t} \quad \hat{f}_s(\omega, t) = \hat{f}_s(\omega, 0) \exp(-\omega^2 t)$$

Then $\hat{f}_s(\omega, 0) = (2/\pi) \int_0^\infty u(x, 0) \sin(\omega x) dx = (2/\pi) \int_0^\infty P(x) \sin(\omega x) dx = (2/\omega\pi) [\cos \omega - \cos 2\omega]$

$$\hat{f}_s(\omega, t) = \hat{f}_s(\omega, 0) \exp(-\omega^2 t) = (2/\omega\pi) [\cos \omega - \cos 2\omega] \exp(-\omega^2 t)$$

Inverse $\hat{f}_s(\omega, t)$ Gives the complete solution

$$u(x, t) = \int_0^\infty \hat{f}_s(\omega, t) \sin(\omega t) d\omega = \int_0^\infty (2/\omega\pi) (\cos(\omega) - \cos(2\omega)) \exp(-\omega^2 t) \sin(\omega x) d\omega$$

Fourier Transform of the Unit-Step Function

- Since

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

using the integration property, it is

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

Properties of the Fourier Transform - Summary

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{ a } X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\sin \omega_0 t$	$x(t) \sin \omega_0 t \leftrightarrow \frac{j}{2} [X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos \omega_0 t$	$x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2} [X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega) d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) ^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Example :

The property of Fourier transform of derivatives can be used for solution of differential equations:

$$y' - 4y = H(t)e^{-4t}$$

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$F\{y'\} - 4F\{y\} = F\{H(t)e^{-4t}\} = \frac{1}{4 + iw}$$

Setting $F\{y(t)\} = Y(w)$, we have

$$iwY(w) - 4Y(w) = \frac{1}{4 + iw}$$

Example :

Then

$$Y(w) = \frac{1}{(4 + iw)(-4 + iw)} = -\frac{1}{16 + w^2}$$

Therefore

$$y(w) = F^{-1}\{Y(w)\} = -\frac{1}{8}e^{-4|t|}$$