

# Kurtosis

# Introduction

The usual kurtosis measure is

$$\beta_2 (F) = \frac{\mu_4 (F)}{\mu_2^2 (F)}$$

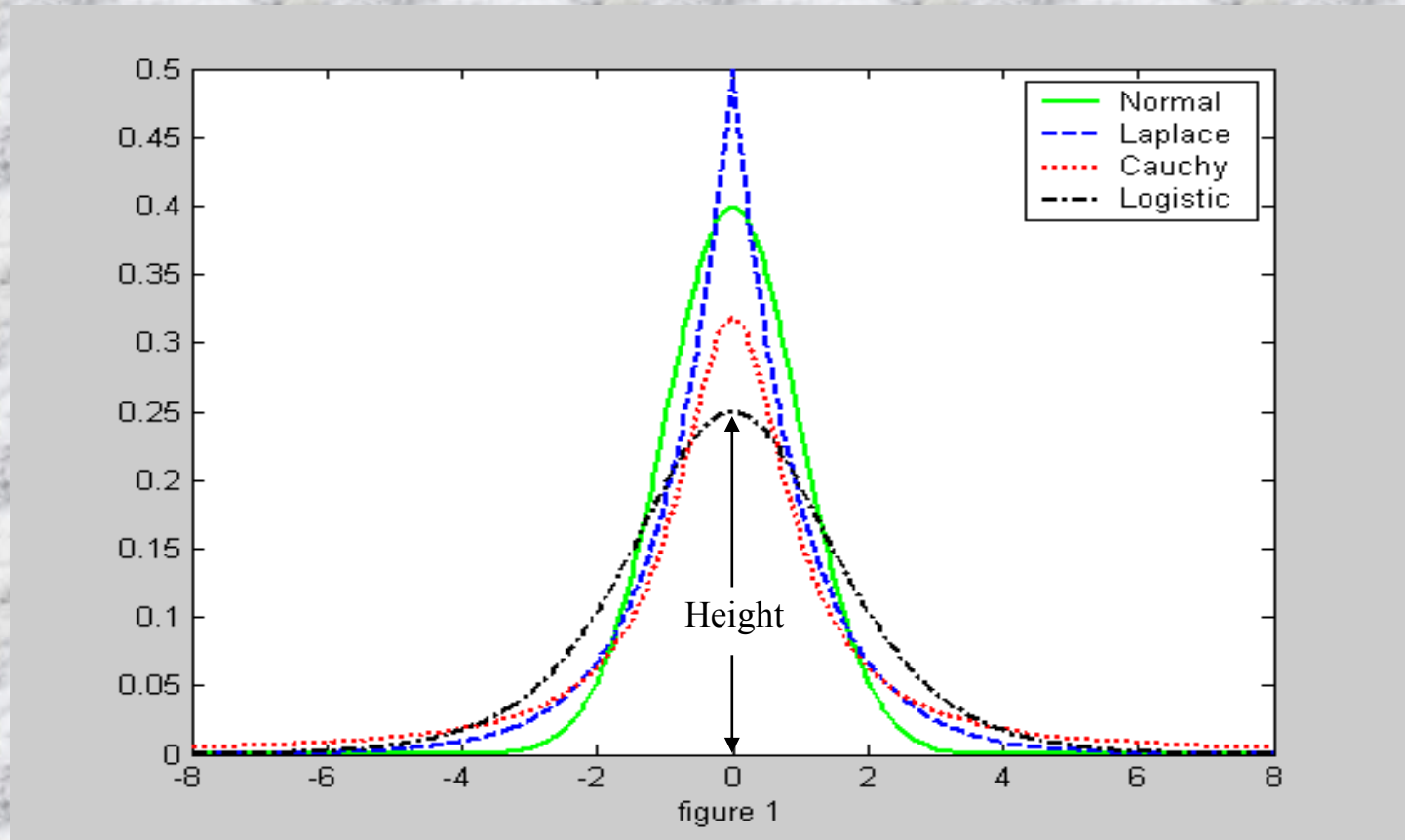
where :

$$\mu_k (F) = E_F (X - E_F (x))^k$$

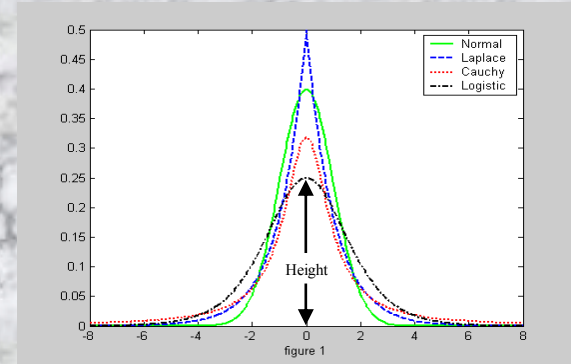
with  $X \sim F(.)$ .

# What is **kurtosis** ?

$\beta_2$  does not sort the distributions based on the height

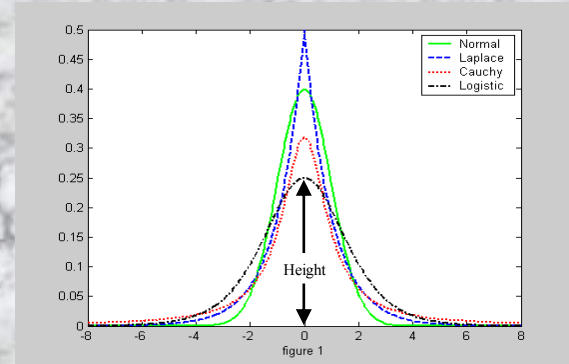


What is **kurtosis** ?



We compute  $\beta_2$  for some distributions, Uniform, Normal, Logistic, Laplace, and Cauchy for the above distributions the kurtosis Measures,  $\beta_2$ , are 1.8, 3, 4.2, 6, and  $\infty$  respectively.

What is **kurtosis** ?



As the height of distributions increases,  
the order of distributions will be  
Logistic, Cauchy, Normal, Laplace, and  
Uniform respectively.

# Disadvantages of $\beta_2$

- 1- It's infinite for heavy tail distributions.
- 2- It doesn't work well for some distributions such as Ali's scale contaminated normal distributions.

$$F_k(x) = \left(1 - \frac{1}{k^2 - 1}\right)\Phi(x) + \frac{1}{k^2 - 1}\Phi\left(\frac{x}{k}\right), \quad k = 2, 3, \dots$$

where  $\Phi(x)$  is the standard normal distribution function.



$$\beta_2 = 3(k^2 + 1)/4 \rightarrow \infty$$

when  $k \rightarrow \infty$

But this sequence converges in distribution to the standard normal distribution as  $k \rightarrow \infty$ .

3- It can be misleading as a departure from normality.

$\beta_2 = 3$  is not a sufficient condition for normality.

# A Modified Measure of Kurtosis

$$\beta_p^q(F) = \frac{E_F[(X - E_F(x))I_{(p,q)}(X)]^4}{E_F^2[(X - E_F(X))^2 I_{(p,q)}(X)]}$$

Where p and q are quantile of order p and q, respectively

with  $X \sim F(.)$ .



# Properties of $\beta_p^q$

P.1.  $\lim_{p \rightarrow 0} \beta_{F^{-1}(p)}^{F^{-1}(1-p)} = \beta_2$

If  $f(x) > 0$  for all  $x$ .

P.2.  $\lim_{p \rightarrow \frac{1}{2}} \beta_{F^{-1}(p)}^{F^{-1}(1-p)} = 0,$

P.3.  $\beta_a^b(F_k) = \beta_a^b(\Phi(x))$   
 $k \rightarrow \infty$

Proof:

since  $\beta_a^b(F_k(x)) = \frac{(1 - \frac{1}{k^2-1}) \int_a^b x^4 \phi(x) dx + A(k)}{\left( (1 - \frac{1}{k^2-1}) \int_a^b x^2 \phi(x) dx + B(k) \right)^2}$

where  $A(k) = \frac{k^5}{k(k^2-1)} \int_{\frac{b}{k}}^{\frac{a}{k}} x^4 \phi(x) dx$

And  $B(k) = \frac{k^3}{k(k^2-1)} \int_{\frac{b}{k}}^{\frac{a}{k}} x^2 \phi(x) dx$

$\lim A(k) = \lim B(k) = 0$  as  $k \rightarrow \infty$

Engineering Mathematics III

We show that the treatment of  $\beta_p^q$  is as the same as  $\beta_2$

# Properties of kurtosis measure

Oja (1981) says a location and scale invariant function  $T$  can be named a kurtosis measure if  $T(G) \geq T(F)$  whenever  $G$  has at least as much Kurtosis as  $F$  according to the definition of relative kurtosis.

$T$  is a kurtosis measure if :

1- It must be location and scale invariant i.e.

$$T(ax+b) = T(x) \quad \text{for } a > 0$$

2- It must preserve one of the orderings.

Ordering  $\ll$  were defined in such a way that  $F \ll G$  means, in some location and scale free, that  $G$  has at least as much mass in the center and tails as  $F$  i.e.

$$\text{if } F \ll_s G \text{ then } T(F) \leq T(G)$$