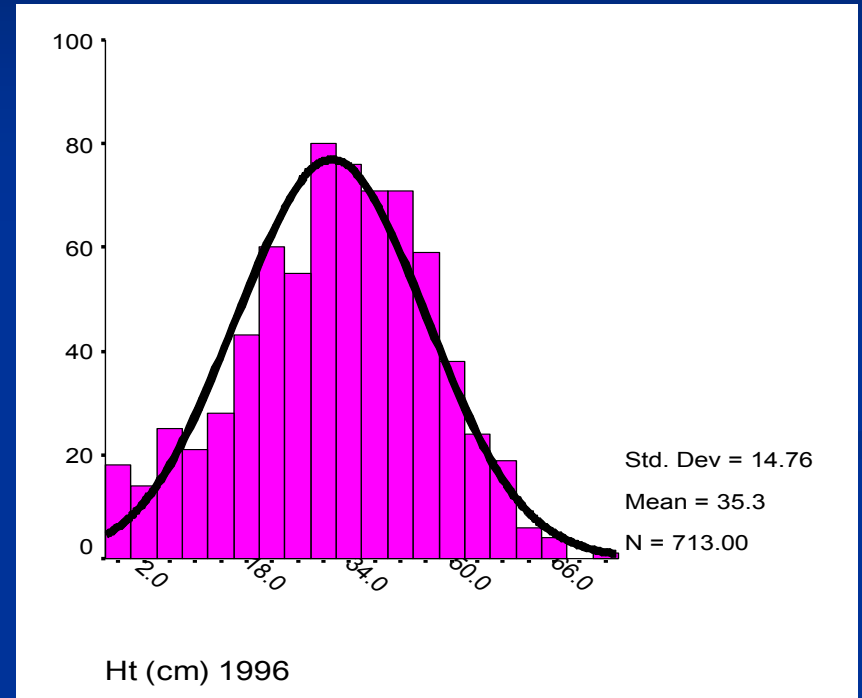


The Binomial, Poisson, and Normal Distributions

Probability distributions

- We use probability distributions because they work—they fit lots of data in real world



Height (cm) of *Hypericum cumulicola* at Archbold Biological Station

Random variable

- The mathematical rule (or function) that assigns a given numerical value to each possible outcome of an experiment in the sample space of interest.

Types of Random variables

- Discrete random variables
- Continuous random variables

The Binomial Distribution

Bernoulli Random Variables

- Imagine a simple trial with only two possible outcomes
 - Success (S)
 - Failure (F)
- Examples
 - Toss of a coin (heads or tails)
 - Sex of a newborn (male or female)
 - Survival of an organism in a region (live or die)



Jacob Bernoulli (1654-1705)

The Binomial Distribution

Overview

- Suppose that the probability of success is p
- What is the probability of failure?
 - $q = 1 - p$
- Examples
 - Toss of a coin ($S = \text{head}$): $p = 0.5 \Rightarrow q = 0.5$
 - Roll of a die ($S = 1$): $p = 0.1667 \Rightarrow q = 0.8333$
 - Fertility of a chicken egg ($S = \text{fertile}$): $p = 0.8 \Rightarrow q = 0.2$

The Binomial Distribution

Overview

- Imagine that a trial is repeated n times
- Examples
 - A coin is tossed 5 times
 - A die is rolled 25 times
 - 50 chicken eggs are examined
- Assume p remains constant from trial to trial and that the trials are statistically independent of each other

The Binomial Distribution

Overview

- What is the probability of obtaining x successes in n trials?
- Example
 - What is the probability of obtaining 2 heads from a coin that was tossed 5 times?

$$P(HHTTT) = (1/2)^5 = 1/32$$

The Binomial Distribution

Overview

- But there are more possibilities:

HHTTTT

HTHTTT

HTTHTT

HTTTTH

THHTTT

THTHTT

THTTTH

TTHHTT

TTHHTH

TTTHHH

$$P(2 \text{ heads}) = 10 \times 1/32 = 10/32$$

The Binomial Distribution

Overview

- In general, if trials result in a series of success and failures,

FFSFFFFSFSFSFFFFFSF...

Then the probability of x successes in that order is

$$\begin{aligned}P(x) &= q \cdot q \cdot p \cdot q \cdot \dots \\ &= p^x \cdot q^{n-x}\end{aligned}$$

The Binomial Distribution

Overview

- However, if order is not important, then

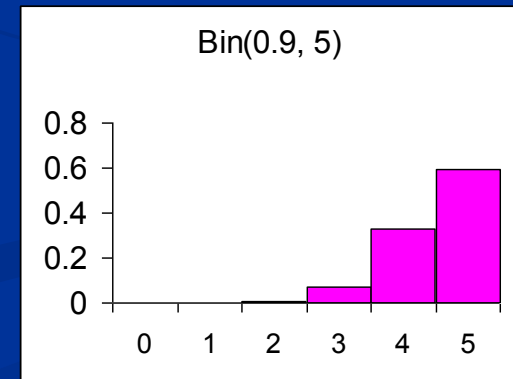
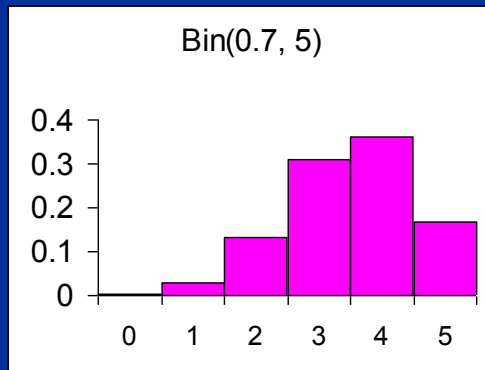
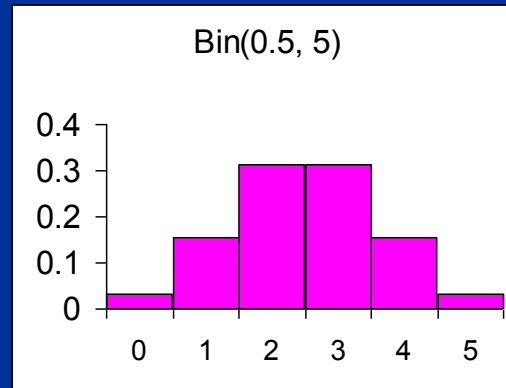
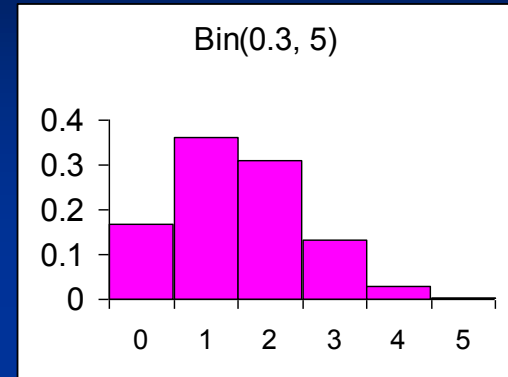
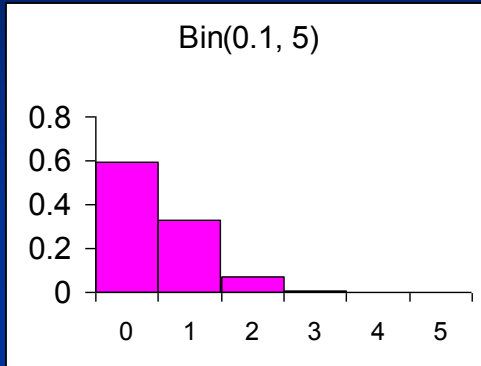
$$P(x) = \frac{n!}{x!(n-x)!} p^x \cdot q^{n-x}$$

where $\frac{n!}{x!(n-x)!}$ is the number of ways to obtain x successes

in n trials, and $i! = i \cdot (i-1) \cdot (i-2) \cdot \dots \cdot 2 \cdot 1$

The Binomial Distribution

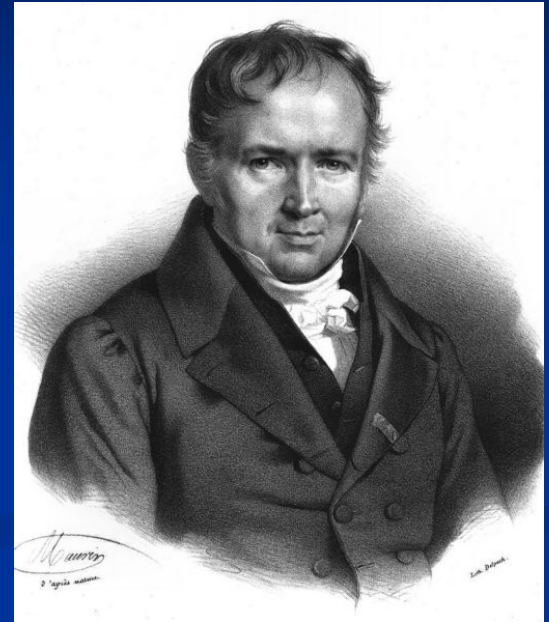
Overview



The Poisson Distribution

Overview

- When there is a large number of trials, but a small probability of success, binomial calculation becomes impractical
 - Example: Number of deaths from horse kicks in the Army in different years
- The mean number of successes from n trials is $\mu = np$
 - Example: 64 deaths in 20 years from thousands of soldiers



Simeon D. Poisson (1781-1840)

The Poisson Distribution

Overview

- If we substitute μ/n for p , and let n tend to infinity, the binomial distribution becomes the Poisson distribution:

$$P(x) = \frac{e^{-\mu} \mu^x}{x!}$$

The Poisson Distribution

Overview

- Poisson distribution is applied where random events in space or time are expected to occur
- Deviation from Poisson distribution may indicate some degree of non-randomness in the events under study
- Investigation of cause may be of interest

The Poisson Distribution

Emission of α -particles

- Rutherford, Geiger, and Bateman (1910) counted the number of α -particles emitted by a film of polonium in 2608 successive intervals of one-eighth of a minute
 - What is n ?
 - What is p ?
- Do their data follow a Poisson distribution?

The Poisson Distribution

Emission of α -particles

- Calculation of μ :

$$\begin{aligned}\mu &= \text{No. of particles per interval} \\ &= 10097/2608 \\ &= 3.87\end{aligned}$$

- Expected values:

$$2680 \times P(x) = 2608 \times \frac{e^{-3.87}(3.87)^x}{x!}$$

No. α -particles	Observed
0	57
1	203
2	383
3	525
4	532
5	408
6	273
7	139
8	45
9	27
10	10
11	4
12	0
13	1
14	1
Over 14	0
Total	2608

The Poisson Distribution

Emission of α -particles

No. α -particles	Observed	Expected
0	57	54
1	203	210
2	383	407
3	525	525
4	532	508
5	408	394
6	273	254
7	139	140
8	45	68
9	27	29
10	10	11
11	4	4
12	0	1
13	1	1
14	1	1
Over 14	0	0
Total	2608	2680

The Poisson Distribution

Emission of α -particles



Random events

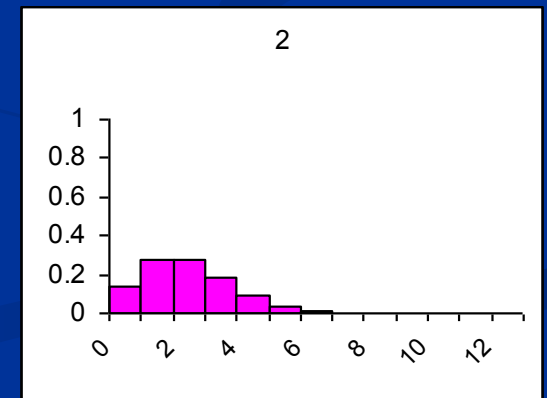
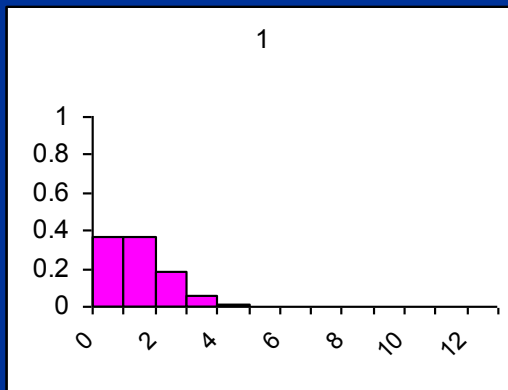
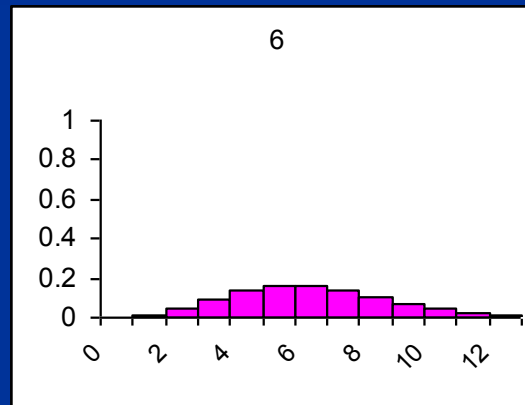
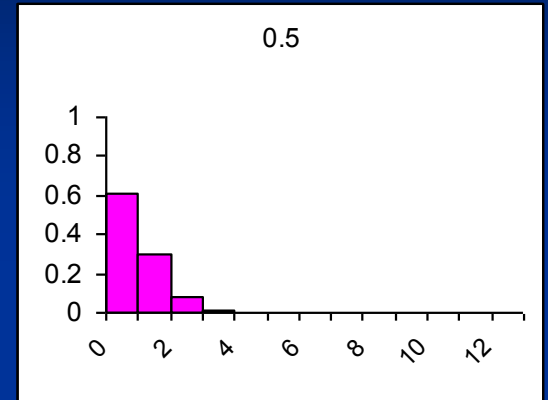
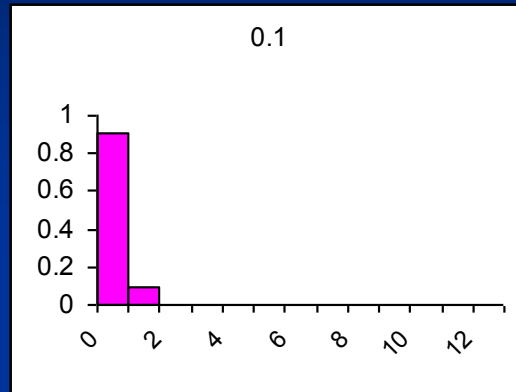


Regular events



Clumped events

The Poisson Distribution



The Expected Value of a Discrete Random Variable

$$E(X) = \sum_{i=1}^n a_i p_i = a_1 p_1 + a_2 p_2 + \dots + a_n p_n$$

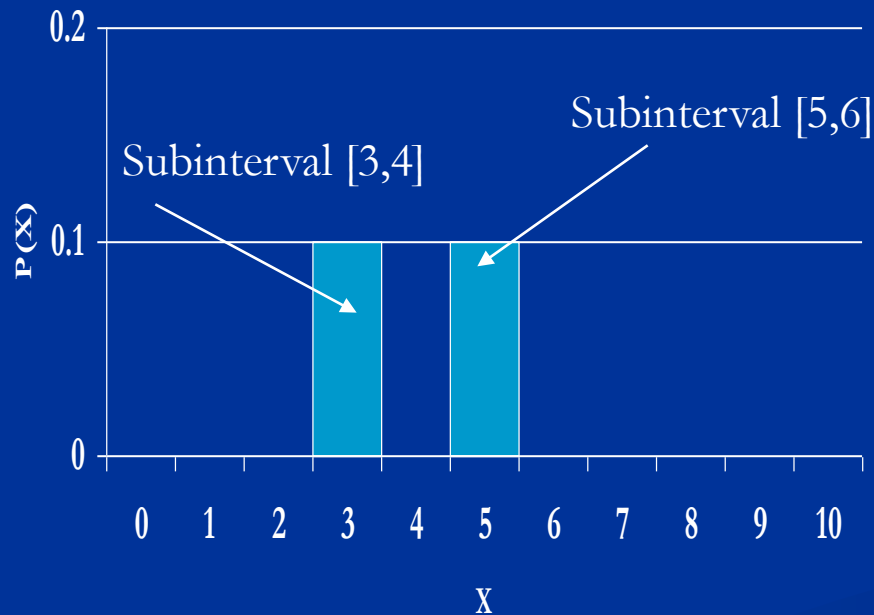
The Variance of a Discrete Random Variable

$$\sigma^2(X) = E[X - E(X)]^2$$

$$= \sum_{i=1}^n p_i \left(a_i - \sum_{i=1}^n a_i p_i \right)^2$$

Uniform random variables

- The closed unit interval, which contains all numbers between 0 and 1, including the two end points 0 and 1



$$f(x) = \begin{cases} 1/10, & 0 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

The probability
density function

The Expected Value of a continuous Random Variable

$$E(X) = \int x f(x) dx$$

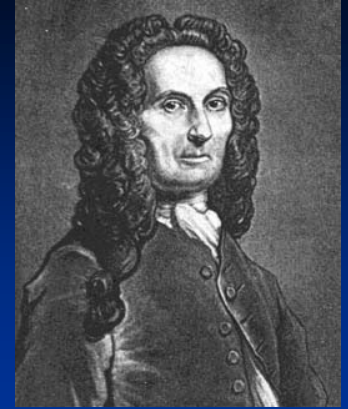
For an uniform random variable x , where $f(x)$ is defined on the interval $[a, b]$, and where $a < b$,

$$E(X) = (b + a) / 2 \quad \text{and} \quad \sigma^2(X) = \frac{(b - a)^2}{12}$$

The Normal Distribution

Overview

- Discovered in 1733 by de Moivre as an approximation to the binomial distribution when the number of trials is large
- Derived in 1809 by Gauss
- Importance lies in the Central Limit Theorem, which states that the sum of a large number of independent random variables (binomial, Poisson, etc.) will approximate a normal distribution
 - Example: Human height is determined by a large number of factors, both genetic and environmental, which are additive in their effects. Thus, it follows a normal distribution.



Abraham de Moivre
(1667-1754)



Karl F. Gauss
(1777-1855)

The Normal Distribution

Overview

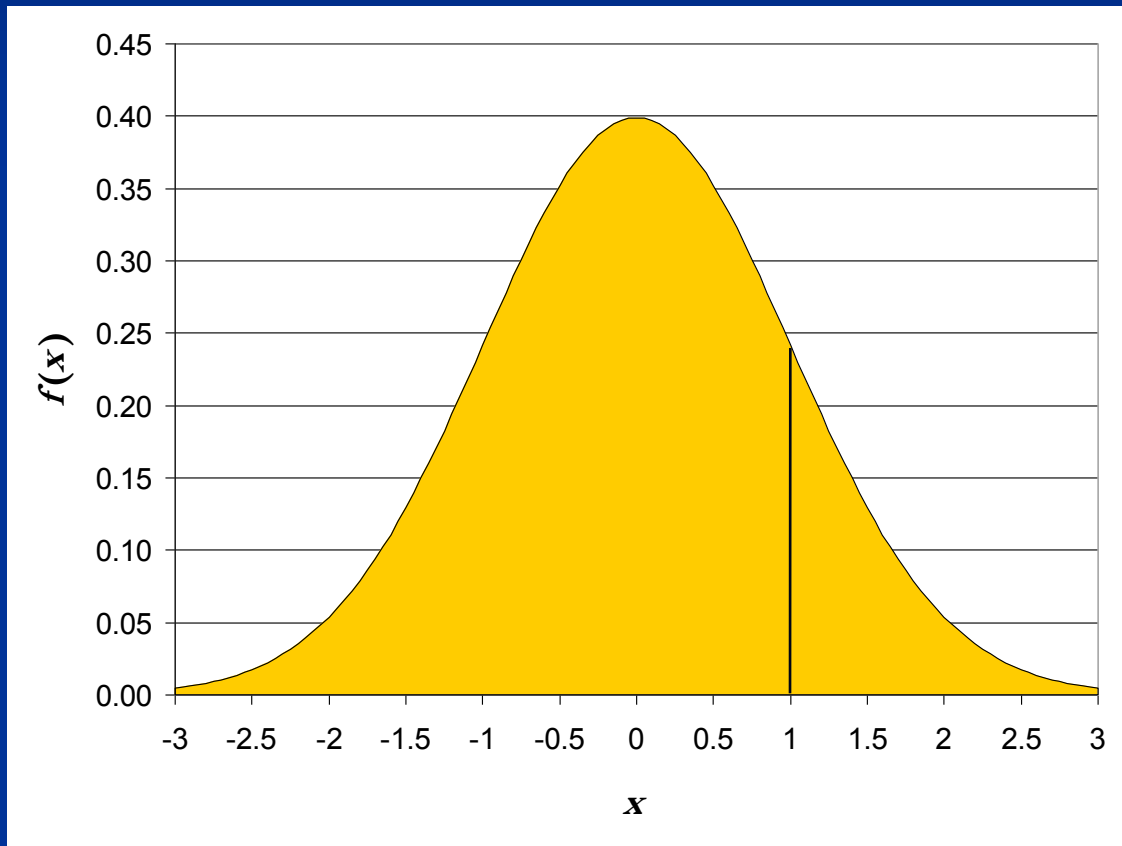
- A continuous random variable is said to be normally distributed with mean μ and variance σ^2 if its probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $f(x)$ is not the same as $P(x)$
 - $P(x)$ would be 0 for every x because the normal distribution is continuous
 - However, $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f(x) dx$

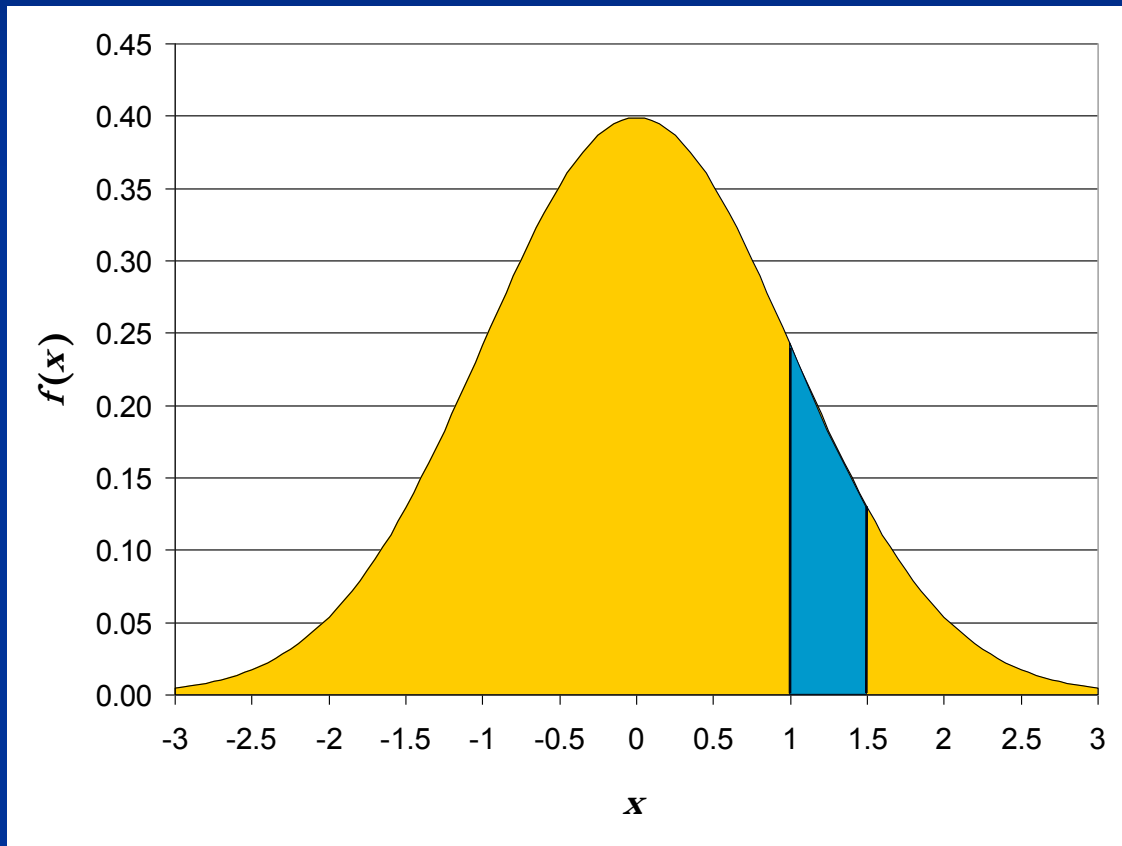
The Normal Distribution

Overview



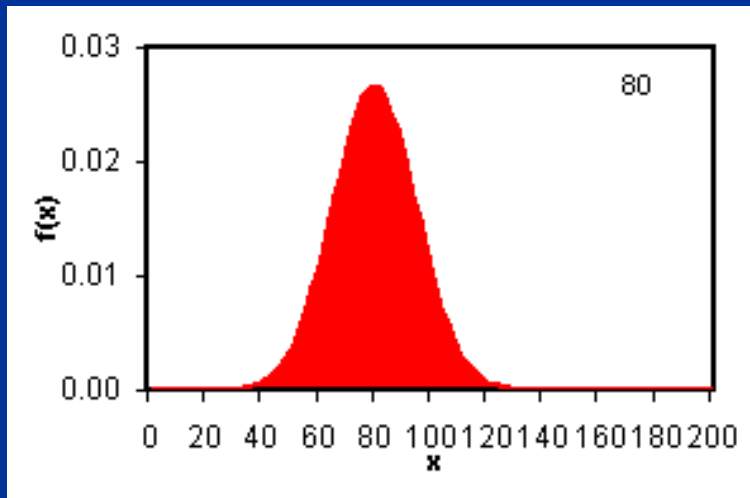
The Normal Distribution

Overview

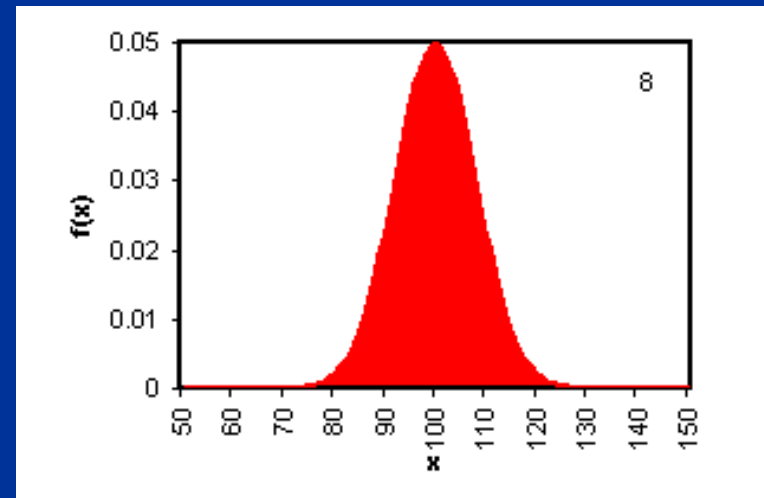


The Normal Distribution

Overview



Mean changes



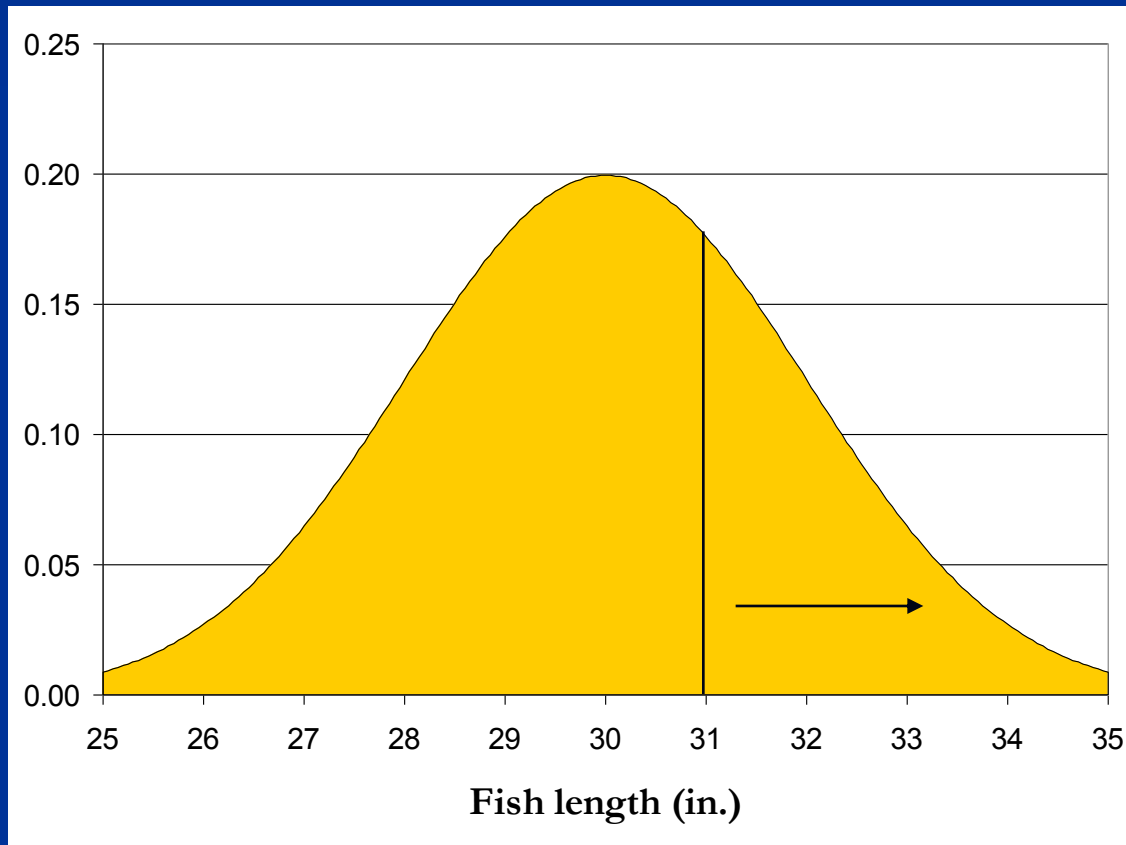
Variance changes

The Normal Distribution

- A sample of rock cod in Monterey Bay suggests that the mean length of these fish is $\mu = 30$ in. and $\sigma^2 = 4$ in.
- Assume that the length of rock cod is a normal random variable
- If we catch one of these fish in Monterey Bay,
 - What is the probability that it will be at least 31 in. long?
 - That it will be no more than 32 in. long?
 - That its length will be between 26 and 29 inches?

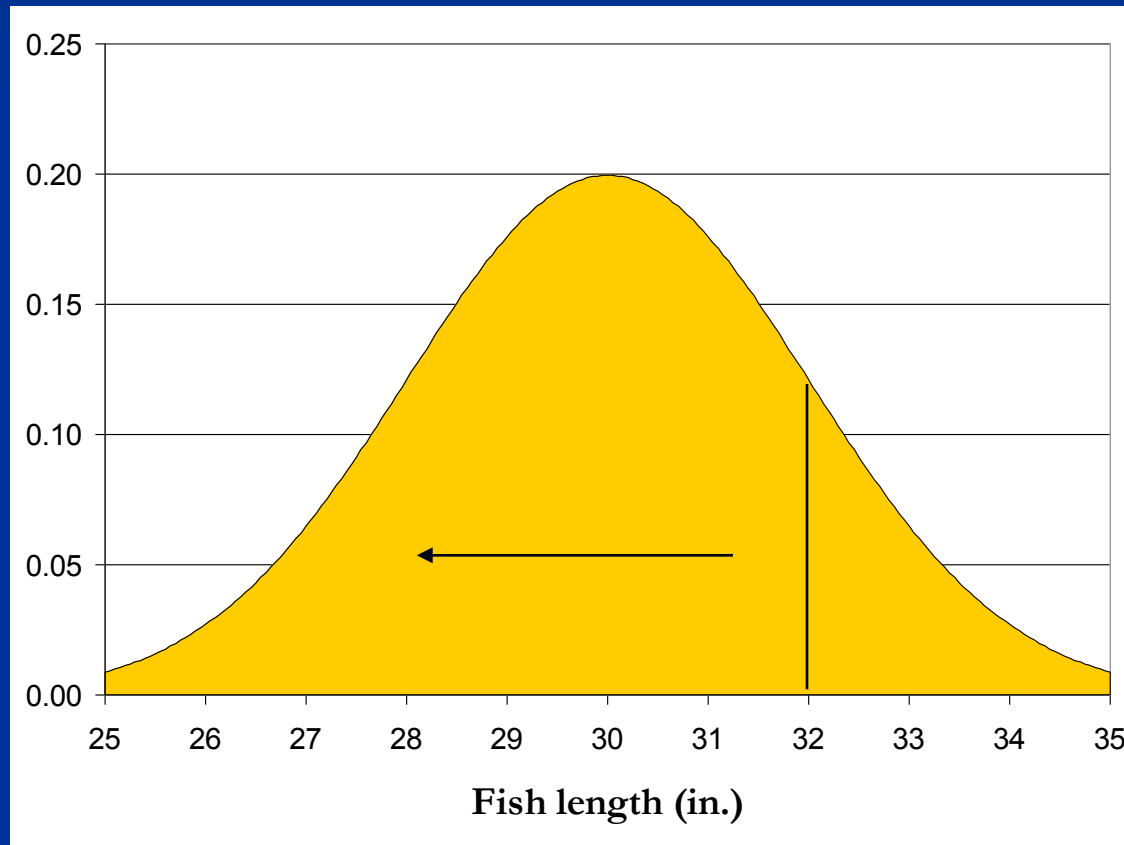
The Normal Distribution

- What is the probability that it will be at least 31 in. long?



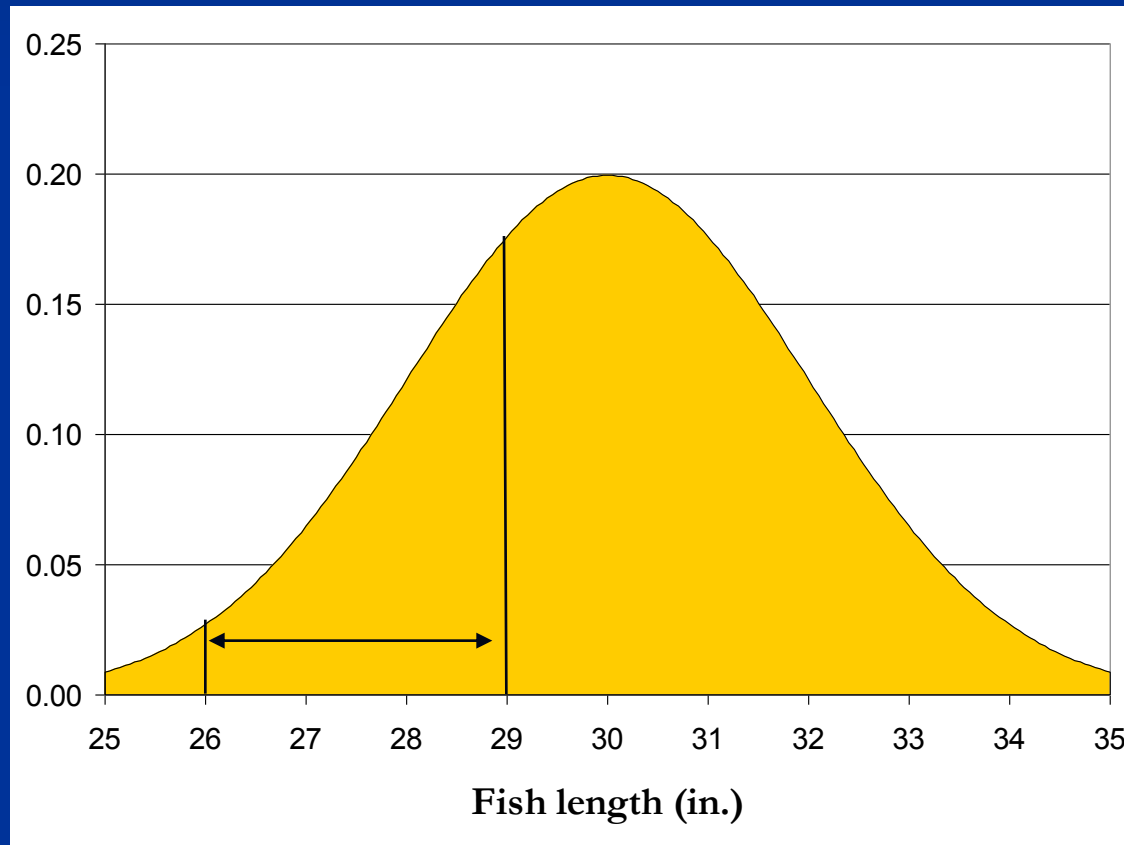
The Normal Distribution

- That it will be no more than 32 in. long?



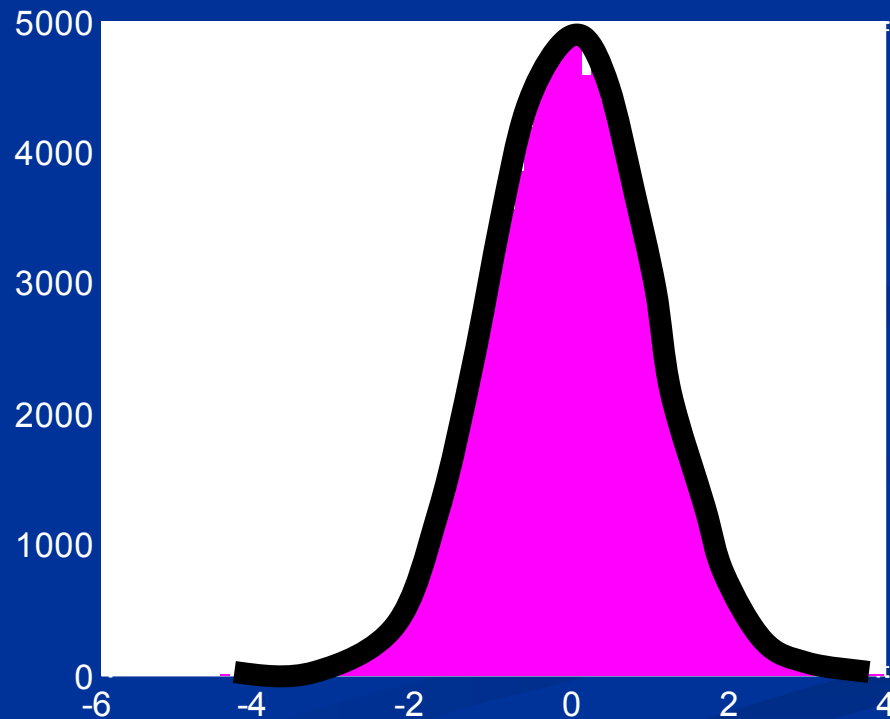
The Normal Distribution

- That its length will be between 26 and 29 inches?



Standard Normal Distribution

- $\mu=0$ and $\sigma^2=1$



Useful properties of the normal distribution

1. The normal distribution has useful properties:
 - Can be added $E(X+Y) = E(X) + E(Y)$ and $\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y)$
 - Can be transformed with *shift* and *change of scale* operations

Consider two random variables X and Y

Let $X \sim N(\mu, \sigma)$ and let $Y = aX + b$ where a and b are constants

Change of scale is the operation of multiplying X by a constant “ a ” because one unit of X becomes “ a ” units of Y .

Shift is the operation of adding a constant “ b ” to X because we simply move our random variable X “ b ” units along the x-axis.

If X is a normal random variable, then the new random variable Y created by these operations on X is also a random normal variable

For $X \sim N(\mu, \sigma)$ and $Y = aX + b$

- $E(Y) = a\mu + b$
- $\sigma^2(Y) = a^2 \sigma^2$
- A special case of a change of scale and shift operation in which $a = 1/\sigma$ and $b = -1(\mu/\sigma)$
- $Y = (1/\sigma)X - \mu/\sigma$
- $Y = (X - \mu)/\sigma$ gives
- $E(Y) = 0$ and $\sigma^2(Y) = 1$

The Central Limit Theorem

- That Standardizing any random variable that itself is a sum or average of a set of independent random variables results in a new random variable that is nearly the same as a standard normal one.
- The only caveats are that the sample size must be large enough and that the observations themselves must be independent and all drawn from a distribution with common expectation and variance.