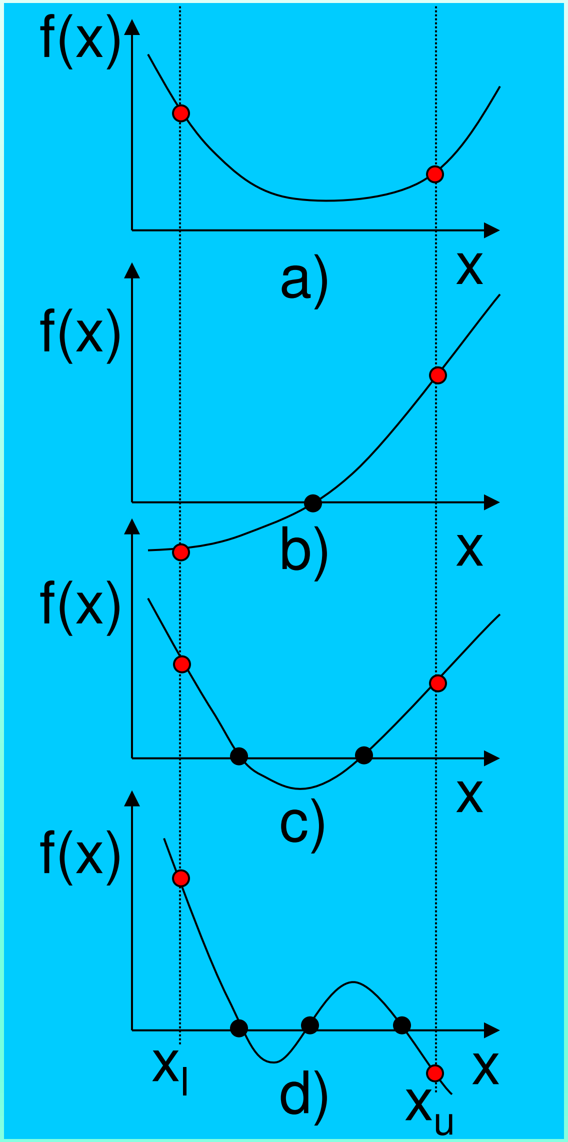
Rate of convergence

Two Fundamental Approaches

1. ***Bracketing or Closed Methods***
   - Bisection Method
   - False-position Method (*Regula falsi*).
2. ***Open Methods***
   - Newton-Raphson Method
   - Secant Method
   - Fixed point Methods

Bracketing Methods



In Figure a) we have the case of $f(x_l)$ and $f(x_u)$ with the same sign, and there is no root in the interval (x_l,x_u).

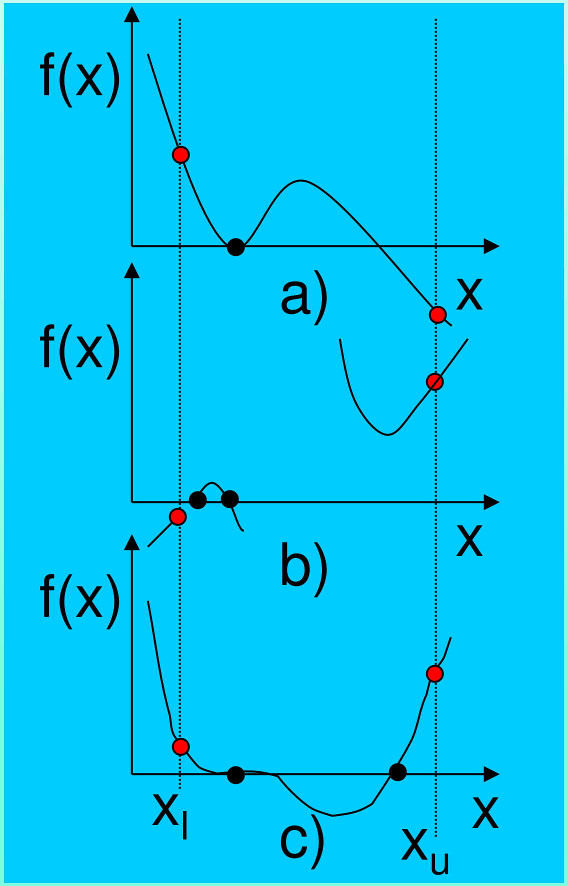
In Figure b) we have the case of $f(x_l)$ and $f(x_u)$ With different sign, and there is a root in the interval (x_l,x_u).

In Figure c) we have the case of $f(x_l)$ and $f(x_u)$ with the same sign, and there are two roots.

In Figure d) we have the case of $f(x_l)$ and $f(x_u)$ with different sign, and there is an odd number of roots.

Bracketing Methods

- Though the cases above are generally valid, there are cases in which they do not hold.



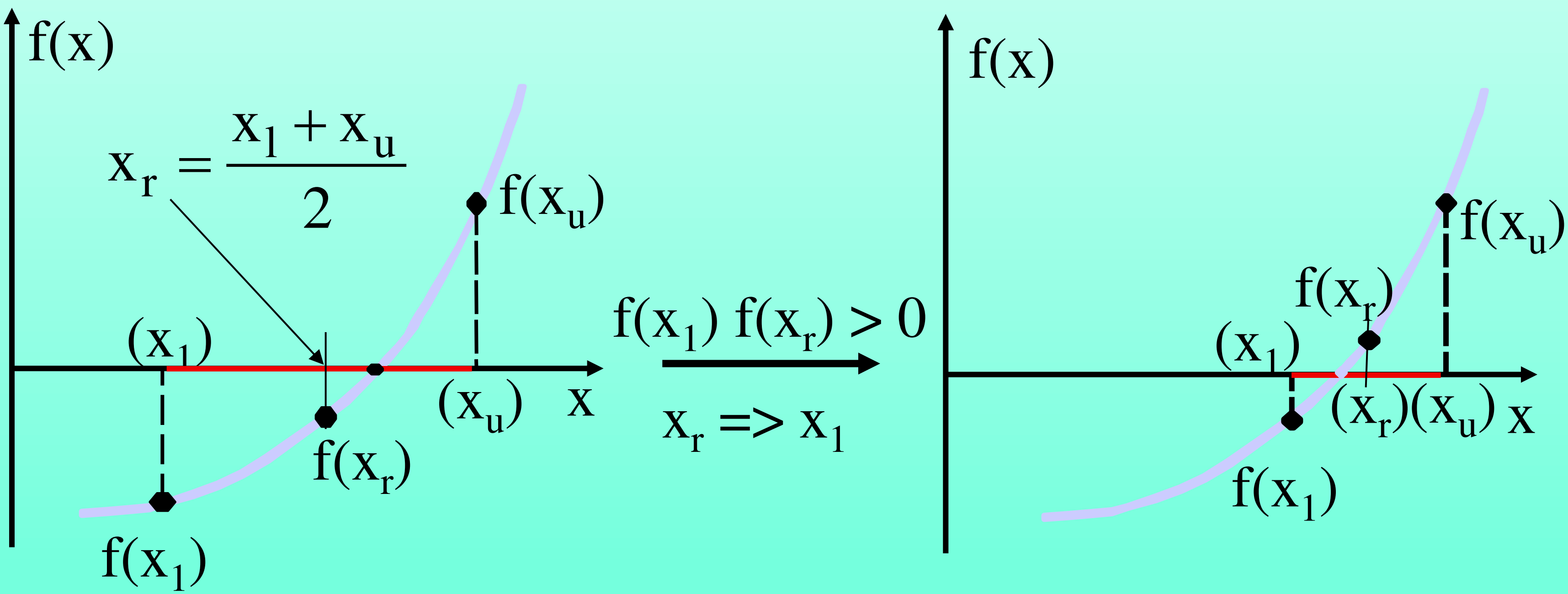
In Figure a) we have the case of $f(x_l)$ and $f(x_u)$ with different sign, but there is a double root.

In Figure b) We have the case of $f(x_l)$ and $f(x_u)$ With different sign, but there are two discontinuities.

In Figure c) we have the case of $f(x_l)$ and $f(x_u)$ with the same sign, but there is a multiple root.

Bracketing Methods (Bisection method)

Bisection Method



Bisection Method

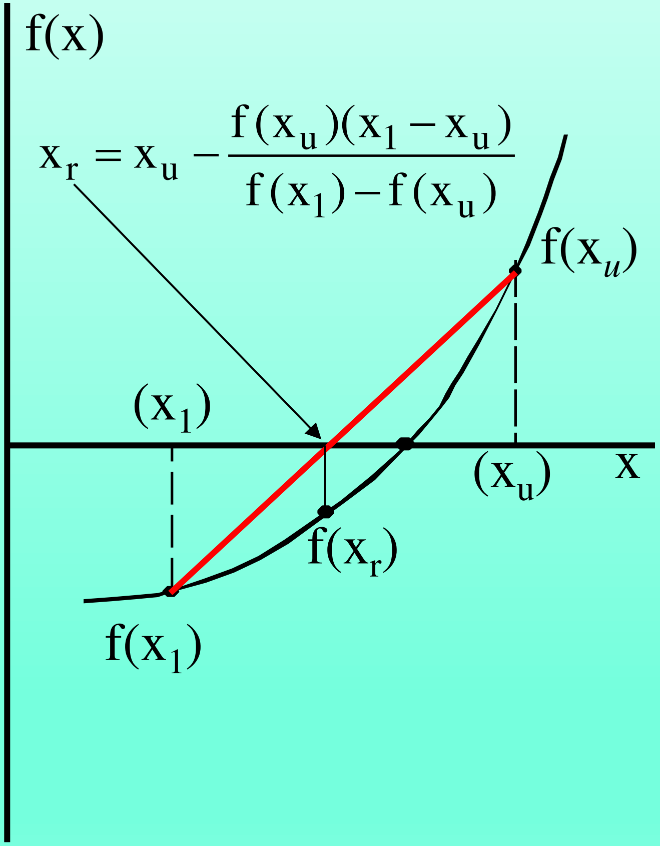
Advantages:

1. Simple
2. Estimate of maximum error: $\left|E_{max}\right| \le \left|\frac{x_l - x_u}{2}\right|$
3. Convergence guaranteed $\left|E_{max}^{i+1}\right| = 0.5\left|E_{max}^{i}\right|$

Disadvantages:

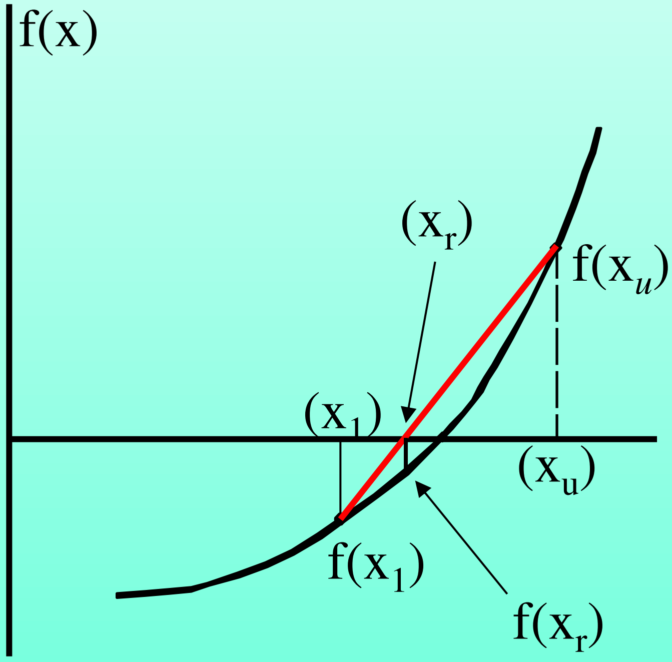
1. Slow
2. Requires two good initial estimates which define an interval around root:
   - use graph of function,
   - incremental search, or
   - trial & error

False-position Method

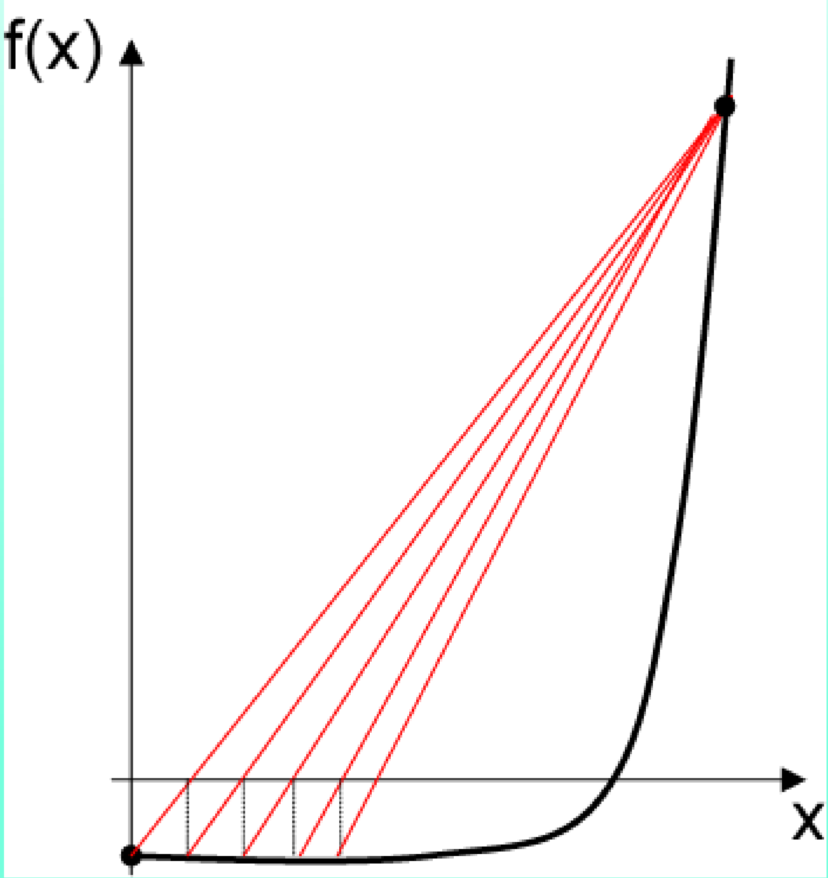


$f(x_1)\, f(x_r) > 0$

$x_1 = x_r$



There are some cases in which the false position method is very slow, and the bisection method gives a faster solution.



Summary of False-Position Method:

Advantages:

1. Simple
2. Brackets the Root

Disadvantages:

1. Can be VERY slow
2. Like Bisection, need an initial interval around the root.

Roots of Equations - Open Methods

Characteristics:

1. Initial estimates need not bracket the root
2. Generally converge faster
3. **NOT** guaranteed to converge

Open Methods Considered:

- Fixed-point Methods
- Newton-Raphson Iteration
- Secant Method

Roots of Equations

Two Fundamental Approaches

1. ***Bracketing or Closed Methods***
   - Bisection Method
   - False-position Method
2. ***Open Methods***
   - One Point Iteration
   - **Newton-Raphson Iteration**
   - **Secant Method**

Newton-Raphson Method:

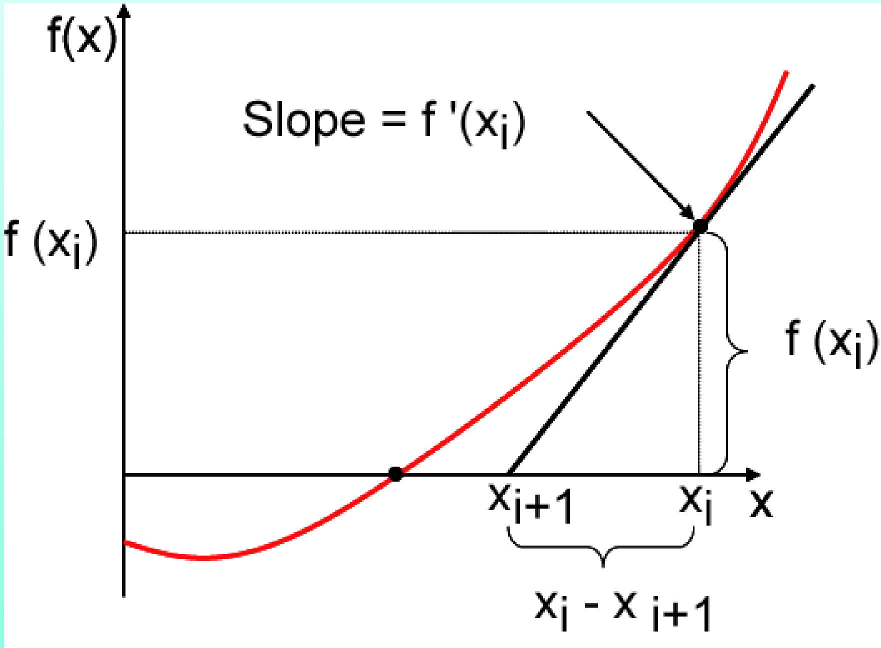
Geometrical Derivation:

Slope of tangent at x_i is

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

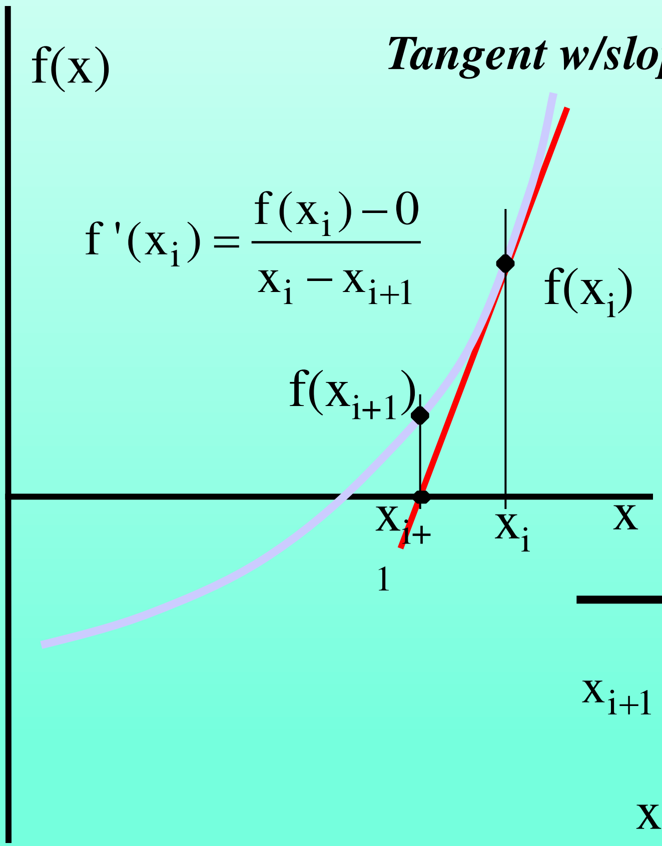
Solve for x_{i+1}:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



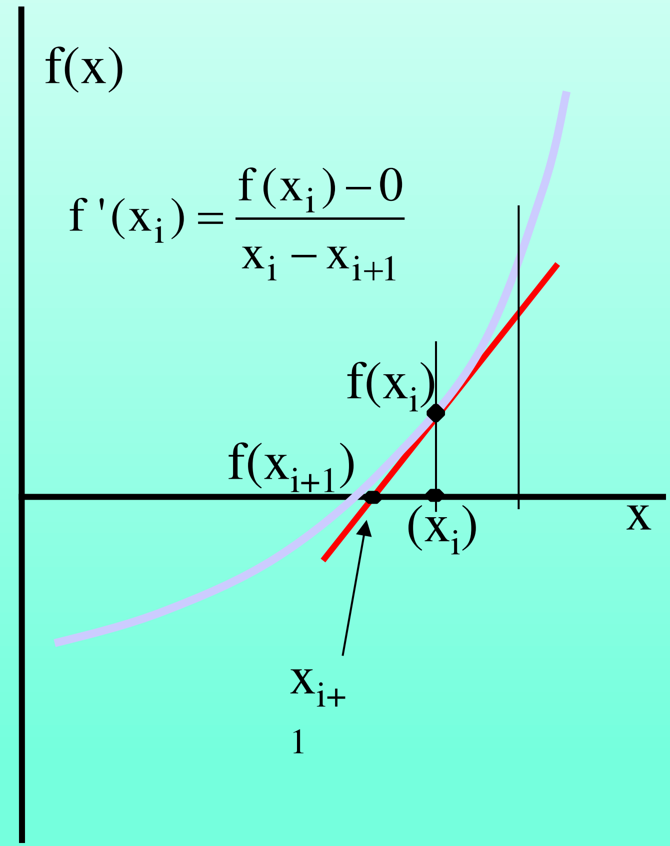
[Note that this is the same form as the generalized one-point iteration, $x_{i+1} = g(x_i)$]

Newton-Raphson Method

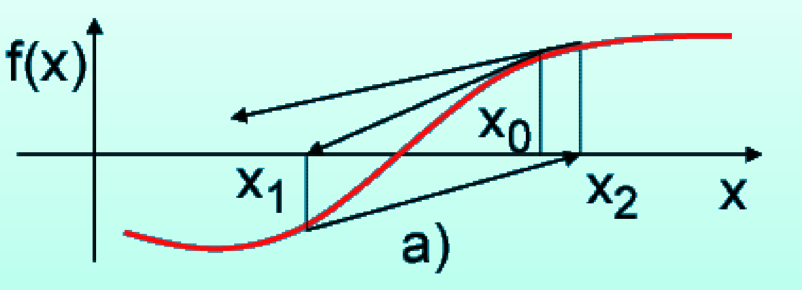


$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_i = x_{i+1}$$

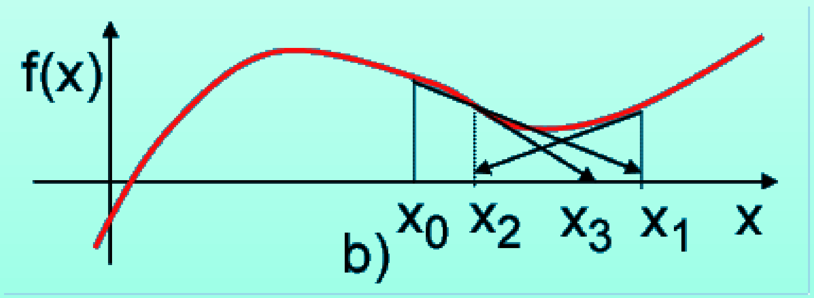


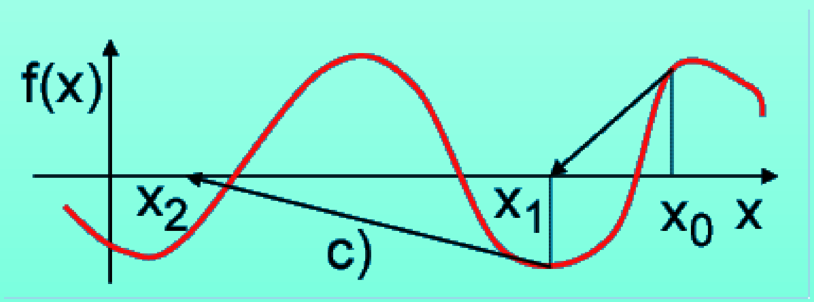
Open Methods



a) Inflection point in the neighboor of a root.

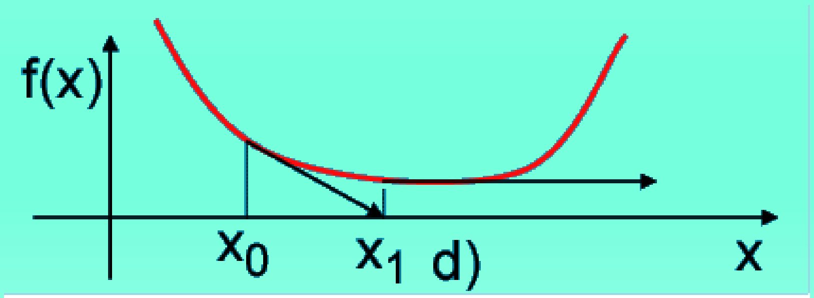
b) Oscilation in the neighboor of a maximum or minimum.





c) Jumps in functions with several roots.

d) Existence of a null derivative.



Open Methods (Newton-Raphson Method)

Bond Example:

To apply Newton-Raphson method to:

$$f(i) = 7,500 - 1,000\left[\frac{1-(1+i)^{-20}}{i}\right] = 0$$

We need the derivative of the function:

$$f\,'(i) = \frac{1,000}{i}\left\{\left[\frac{1-(1+i)^{-20}}{i}\right] - 20(1+i)^{-21}\right\}$$

Rate of convergence

compares the convergence of all the methods.