Zeroes of Transcendental and Polynomial Equations

Bisection method, Regula-falsi method

and Newton-Raphson method

PRELIMINARIES

Solution of equation $f(x) = 0$

A number α (real or complex) is a *root* of the equation $f(x) = 0$ if $f(\alpha) = 0$.

<u>Location of the root</u>

<u>Theorem</u>

If f(x) is a continuous function in the closed interval [a, b] and $f(a) \cdot f(b) < 0$ [i.e. f (a) and f (b) are of opposite signs] then the equation $f(x) = 0$ has at least one real root in the open interval (a, b).

Zero's of a Polynomial and Transcendental Equations:

Given an equation f(x) = 0, where f(x) can be of the forum

(i) $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots + a^n$ Algebraic Polynomial

(ii) or trigonometric, exponential or logarithmic function

$$f(x) = ax + b \log x \text{ i.e. (Transcendental)}$$

if $f(\xi) = 0$ for some ξ, than $x= \xi$ is said to be a zero is a root of multiplicity p if $f(x) = (x- \xi)^p g(x) = 0$ where $g(\xi) \neq 0$.

We can define the root of an equation as the value of x that makes

f(x) = 0. The roots are some limites called the zeros of the equation. There are many functions for which the root cannot be determined so easily.

One method to obtain an approx. solution is to plot the function and determine where it crosses X-axis. Graphical methods provide rough estimates of roots and lack precision.

The standard methods for locating roots typically fall into some what related but primarily distinct problem areas

How to locate the real roots of $p_n(x) = 0$, where

$p_n(x) = a_0x^n + a_1x^{n-1} + ...+ a_n$.

I. The Number of positive roots of $p_n(x) = 0$, where the coefficients 'a's are real cannot exceed the number of changes in signs of the coefficients in the polynomial $p_n(x)$ and the number of negative roots of $p_n(x)$ cannot exceed the number of changes of the sign of the coefficients in $p_n(-x) = 0$

II. Largest rot of $p_n(x) = 0$ is approximately equal to the root of a0x+a1=0. The smallest roots of pn(x)=0 may be approximated by $a_{n-1}x + a_n = 0$

III. If p(a) and p(b) have opposite signs, then there are odd number of real roots of $p_n(x) = 0$ between (a,b). If p(a) + p(b) have the same sign than there are no or an even number of real roots between a and h.

BISECTION METHOD

- Locate the interval (a, b) in which root lies.
- Bisect the interval (a, b).
- Choose the half interval in which the root lies.
- Bisect the half interval.
- Repeat the process until the root converges.

Example

- Find the root of the equation $x^3 - x - 1 = 0$ by bisection method.

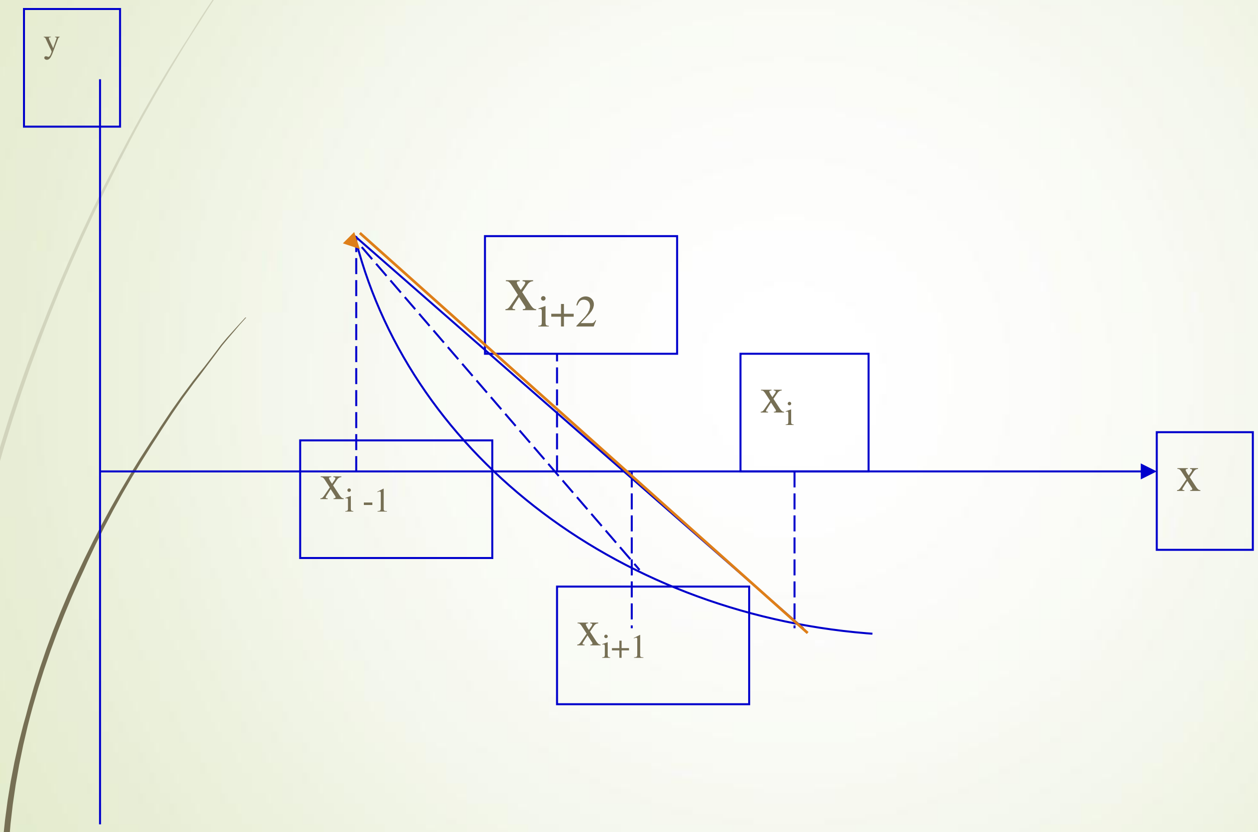
- **Solution**

  - $f(1).\ f(2) < 0$

    Hence a root lies in the interval (1,2).
  - Bisecting , we get two half intervals (1,1.5) and (1.5,2).

    The root lies in (1,1.5).
  - Repeat the process until the root converges to **1.3247**.

Regula Falsi Method



REGULA FALSI METHOD

- Locate the interval (a, b) in which the root lies.
- First Approximation to the root

$$X_0 = \frac{[a * f(b) - b * f(a)]}{[f(b) - f(a)]}$$

- Locate the next interval (a, X_0) or (X_0, b) in which root lies.
- Repeat the process until the root converges.

Example

- Find the real root of $x^3 - 9x + 1 = 0$.
- **Solution**
  - A root lies between 2 and 3.
  - Applying Regula Falsi Method iteratively, the root converges to **2.9428** after **4** iterations.

SUCCESSIVE APPROXIMATION METHOD

- Rewrite the equation $f(x) = 0$ in the form

  $x = \phi(x)$.

- Choose the initial approximation X_0
- $x_1 = \phi(x_0)$, $x_2 = \phi(x_1)$,…,
- The sequence of approximations converges to a root if $| \phi' (x) | < 1$ in the interval containing the root α.

Example

- Solve cos x + 3 = 2x

- **Solution**

  - Write $x = \frac{(\cos x + 3)}{2} = f(x)$
  - $\phi'(x) = (-\frac{1}{2}) \sin x$ and $|\phi'(x)| < 1$ in $(0, \pi/2)$.
  - Choose $x_0 = \pi/2 = 1.5708$.
  - Successive approximation will yield the root as **1.5236** in 12 th iteration.

NEWTON RAPHSON METHOD

- Locate the interval (a, b).
- Choose a or b which is nearer to the root as the first approximation x_0 to the root.

Next approximation $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Repeat the process until the root converges.

Example

- Solve $e^x - 1 = 2x$
- Solution
  - The root is near 1.
  - Take $x_1 = 1$.
  - $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3922$

$$\frac{f(x_1)}{f'(x_1)}$$

Successive iterations will yield the root **1.2564.**

SECANT METHOD

- In Newton Raphson Method we have to evaluate f'(x) at every iteration.
- In this method f'(x) is approximated by the formula
- $f'(x_n) \simeq \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$
- Thus, $x_{n+1} = \dfrac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

Example

- Find a root of $x^3 - 4x + 1 = 0$ using secant method.

- **Solution**

  - The root lies in (0,1) as $f(0) = 1$, $f(1) = -2$.

  - Successive application of secant formula yields the root **0.2541.**

Comparison

Method	Iterative formula	Order of convergence	Evaluation of functions for iteration	Reliability of convergence
Bisection	$x_{n+1} = \frac{x_n + x_{n-1}}{2}$	Gain of one bit per iteration	1	Guaranteed convergence
False position	$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$	1	1	Guaranteed convergence
Successive approximation	$x_n = \phi(x_{n-1})$	1	1	Easy to programme.No guaranteed convergence
Newton raphson	$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n)}$	2	2	Convergence depends on stating value. Fast convergence
Secant	$x_{n+1} = \frac{x_{n-1}\ f(x_n)-x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$	1.62	1	No guarantee if not near the root

GRAEFFES' ROOT SQUARING METHOD

- This is a direct method.
- This method is used to find **all** the roots of a polynomial equation with real coefficients.

- For any *n*th degree polynomial equation the following results will apply.
  - **Results**
    - There will be *n* roots for an *n*th degree polynomial equation.
    - There will be at least one real root if *n* is odd.
    - Complex roots occur only in pairs.
    - Descartes' rule of signs will be true.

Descartes' Rule of Signs

- Number of positive roots of $f(x) = 0$ is equal to the number of sign changes of the coefficients or is less than this number by an even integer.
- The number of negative roots of $f(x) = 0$ is obtained by considering the number of sign changes in $f(-x)$.

Root Squaring

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \Lambda + a_n = 0$

Then $f(x)\, f(-x) = a_0^2 x^{2n} - (a_1^2 - 2a_0 a_2)\, x^{2n-2} + (a_2^2 - 2a_1 a_3 + 2a_0 a_4) x^{2n-4} + \Lambda + (-1)^n\, a_n^2 = 0.$

Substitute $y = -x^2$ the equation reduces to

$y^n + C_1\, y^{n-1} + C_2\, y^{n-2} + \Lambda + C_{n-1}\, y + C_n = 0$

where $C_1 = a_1^2 - 2a_0 a_2$, $C_2 = a_2^2 - 2a_1 a_3 + 2a_0 a_4$, … $C_n = a_n^2$

The derived polynomial is of the same degree as the original polynomial and its roots are $-x_1^2$, $-x_2^2$, …, $-x_n^2$ where x_1, x_2, …, x_n are the roots of the original polynomial.

If we apply this root squaring process repeatedly it will yield successive derived polynomial having roots which are negative of successively higher even powers (2*k* after *k* squaring) of the roots of original polynomial.

Root Squaring

After *k squaring* let the reduced polynomial be

$y^n + b_1 y^{n-1} + b_2 y^{n-2} + \Lambda + b_n = 0$.Then the roots of this polynomial are:

$x_1^m, -x_2^m, \ldots, -x_n^m$ where $m = 2^k$.

Let $R_i = -x_i^2$, $i = 1, 2, \ldots, n$. Assuming $|x_1| > |x_2| > \Lambda > |x_n|$
we get $|R_1| >> |R_2| >> |R_3| \ldots >> |R_n|$

Hence $-b_1 = \sum R_i = R_1$

$$b_2 = \sum R_i R_j \simeq R_1 R_2$$

$$-b_3 = \sum R_i R_j R_k \simeq R_1 R_2 R_3$$

$$(-1)^n b_n = R_1 R_2 \ldots R_n$$

We get $R_i = -\dfrac{b_i}{b_{i-1}}$ $i = 1, 2, \ldots, n$ and $|R_i| = \dfrac{|b_i|}{|b_{i-1}|}$

Fix sign using Descartes' rule and actual substituition.

Example

Solve $x^3 - 2x^2 - 5x + 6 = 0$ using Root squaring method

Solution

	k	2^k	Coefficients			
			a_0	a_1	a_2	a_3
	0	1	1	-2	-5	6
			1	4 10	25 24	36
First squaring	1	2	1	14	49	36
			1	196-98	2401-1008	1296
Second squaring	2	4	1	98	1393	1296
			1	9604-2786	1940449- 254016	1679616
Third squaring	**3**	**8**	**1**	**6818**	**1686433**	**1679616**

Squaring

$$|x_1| = |b_1|^{1/8} = (6818)^{1/8} = 3.0144$$

$$|x_2| = \left|\frac{b_2}{b_1}\right|^{1/8} = \left|\frac{1686433}{6818}\right|^{1/8} = 1.9914$$

$$|x_3| = \left|\frac{b_3}{2}\right|^{1/8} = \left|\frac{1679616}{1686433}\right|^{1/8} = 0.9995$$

By Descartes' rule of signs and by actual substitution, we get

the roots as 3.01443, -1.9914, 0.9995.

The exact roots are 3, -2, 1.