Matrix Decomposition methods

LU Decomposition

$A=LU$

$Ax=b \Rightarrow LUx=b$

Define $Ux=y$

$Ly=b$ Solve y by forward substitution

ERO's must be performed on b as well as A

The information about the ERO's are stored in L

Indeed y is obtained by applying ERO's to b vector

$Ux=y$ Solve x by backward substitution

ERO$\Rightarrow$# of rows with at least one nonzero entry

LU Decomposition by Gaussian elimination

There are infinitely many different ways to decompose A.
Most popular one: U=Gaussian eliminated matrix
L=Multipliers used for elimination

$$A=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ m_{2,1} & 1 & 0 & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & 1 & \vdots \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & \cdots & 1 \end{bmatrix}\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1n-1}^{(n)} & a_{n-1n}^{(n)} \\ 0 & 0 & 0 & 0 & a_{nn}^{(n)} \end{bmatrix}$$

Compact storage: The diagonal entries of L matrix are all 1's, they don't need to be stored. LU is stored in a single matrix.

Operation count

- A=LU Decomposition $\frac{n^3}{3} - \frac{n}{3}$ Done only once
- Ly=b forward substitution $\frac{n^2 - n}{2}$
- Ux=y backward substitution $\frac{n^2 + n}{2}$
- Total $\frac{n^3}{3} + n^2 - \frac{n}{3}$
- For different RHS vectors, the system can be efficiently solved.

Pivoting

- Computer uses *finite-precision* arithmetic
- A small error is introduced in each arithmetic operation, *error propagates*
- When the pivotal element is very small, the multipliers will be large.
- Adding numbers of widely differening magnitude can lead to *loss of significance*.
- To reduce error, row interchanges are made to *maximise* the magnitude of *the pivotal element*

Example: Without Pivoting

4-digit arithmetic
$$\begin{bmatrix} 1.133 & 5.281 \\ 24.14 & -1.210 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ 22.93 \end{bmatrix}$$

$$m_{21} = \frac{24.14}{1.133} = 21.31$$

$$\begin{bmatrix} 1.133 & 5.281 \\ 0.000 & -113.7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ -113.8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.9956 \\ 1.001 \end{bmatrix}$$

Loss of significance

Example: With Pivoting

$$\begin{bmatrix} 24.14 & -1.210 \\ 1.133 & 5.281 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 6.414 \end{bmatrix}$$

$$m_{21} = \frac{1.133}{24.14} = 0.04693 \qquad \begin{bmatrix} 24.14 & -1.210 \\ 0.000 & 5.338 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 5.338 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$$

Pivoting procedures

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)}
\end{bmatrix}
$$

Eliminated part

Pivotal row

Pivotal column

Row pivoting

- Most commonly used *partial pivoting* procedure
- Search the pivotal column
- Find the largest element in magnitude
- Then switch this row with the pivotal row

Row pivoting

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)}
\end{bmatrix}
$$

Interchange these rows

Largest in magnitude

Column pivoting

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)}
\end{bmatrix}
$$

Largest in magnitude

Interchange these columns

Complete pivoting

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1i}^{(1)} & \cdots & a_{1j}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2i}^{(2)} & \cdots & a_{2j}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3i}^{(3)} & \cdots & a_{3j}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{ni}^{(i)} & \cdots & a_{nj}^{(i)} & \cdots & a_{nn}^{(i)}
\end{bmatrix}
$$

Interchange these rows

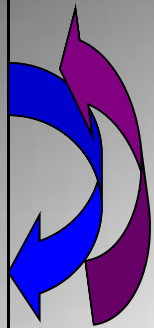
Largest in magnitude

Interchange these columns

Row Pivoting in LU Decomposition

- When two rows of A are interchanged, those rows of b should also be interchanged.
- Use a pivot vector. Initial pivot vector is integers from 1 to n.
- When two rows (i and j) of A are interchanged, apply that to pivot vector.

$$p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ n \end{bmatrix}$$



$$p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ j \\ \vdots \\ i \\ \vdots \\ n \end{bmatrix}$$

Modifying the b vector

- When LU decomposition of A is done, the pivot vector tells the order of rows after interchanges
- Before applying forward substitution to solve Ly=b, modify the order of b vector according to the entries of pivot vector

$$p = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 8 \\ 6 \\ 7 \\ 5 \\ 9 \end{bmatrix} \quad b = \begin{bmatrix} 7.3 \\ 8.6 \\ -1.2 \\ 4.8 \\ 9.6 \\ 5.2 \\ -2.7 \\ 3.5 \\ -6.9 \end{bmatrix} \quad b' = \begin{bmatrix} 7.3 \\ -1.2 \\ 8.6 \\ 4.8 \\ 3.5 \\ 5.2 \\ -2.7 \\ 9.6 \\ -6.9 \end{bmatrix}$$

Example continued...

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3 & 2 \\ 1 & 4 & -2 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Eliminate a_{21} and a_{31} by using a_{11} as pivotal element

A=LU in compact form (in a single matrix)

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3 & 2 \\ -0.25 & 3.5 & -1.75 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Multipliers (L matrix)

Example continued...

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3 & 2 \\ -0.25 & 3.5 & -1.75 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Column search: Maximum magnitude at the third row
Interchange 2nd and 3rd rows

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ -0.25 & 3.5 & -1.75 \\ 0 & 3 & 2 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Example continued...

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ -0.25 & 3.5 & -1.75 \\ 0 & 3 & 2 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Eliminate a_{32} by using a_{22} as pivotal element

$$A' = \begin{bmatrix} -4 & -2 & 1 \\ -0.25 & 3.5 & -1.75 \\ 0 & 3/3.5 & 3.5 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Multipliers (L matrix)

Example continued...

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 3/3.5 & 1 \end{bmatrix} \begin{bmatrix} -4 & -2 & 1 \\ 0 & 3.5 & -1.75 \\ 0 & 0 & 3.5 \end{bmatrix} \quad p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 12 \\ -5 \\ 3 \end{bmatrix} \Rightarrow b' = \begin{bmatrix} -5 \\ 3 \\ 12 \end{bmatrix}$$

A'x=b'   LUx=b'

Ux=y

Ly=b'

Example continued...

Ly=b'

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 3/3.5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 12 \end{bmatrix}$$

Forward substitution

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.75 \\ 10.5 \end{bmatrix}$$

Ux=y

$$\begin{bmatrix} -4 & -2 & 1 \\ 0 & 3.5 & -1.75 \\ 0 & 0 & 3.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.75 \\ 10.5 \end{bmatrix}$$

Backward substitution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Different forms of LU factorization

- Doolittle form

  Obtained by

  Gaussian elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Crout form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

- Cholesky form

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$