Trapezoidal rule

Trapezoid Rule

- Simplest way to approximate the area under a curve – using **first order polynomial** (a straight line)
- Using Newton's form of the interpolating polynomial:

$$p_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

- Now, solve for the integral:

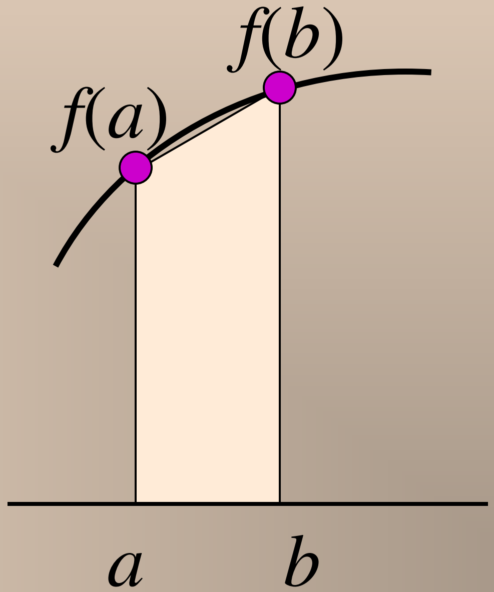
$$I = \int_a^b f(x)dx \approx \int_a^b p_1(x)dx$$

Trapezoid Rule

$$I \approx \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

Trapezoid Rule

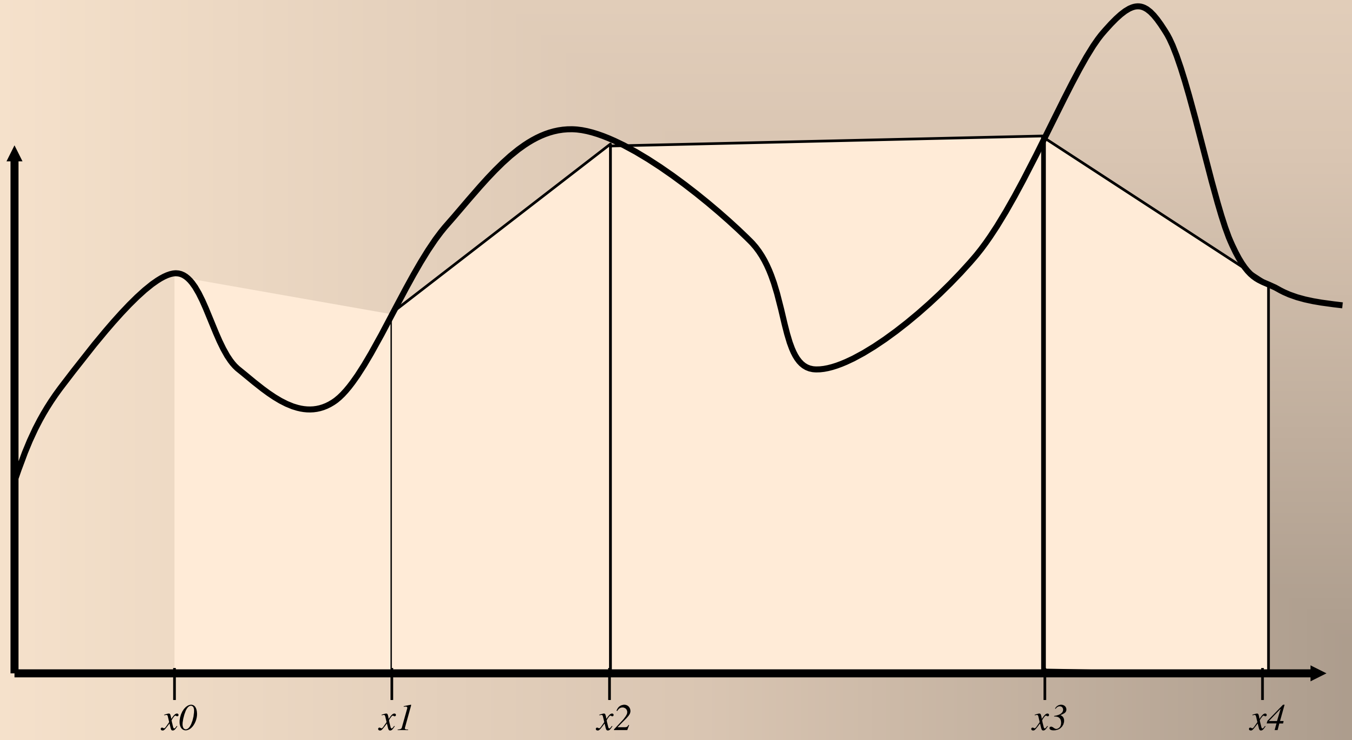
$$\boxed{I \approx \frac{(b - a)}{2}\left[f(a) + f(b) \right]}$$



$$I \approx \text{width} \times \text{average height}$$

Trapezoid Rule

- Improvement?



Trapezoid Rule Error

- The integration error is:

$$E_t = -\frac{1}{12} f''(\xi) h^3 = -\frac{(b-a)}{12} f''(\xi) h^2 \qquad O(h^3)$$

- Where $h = b - a$ and ξ is an unknown point where $a < \xi < b$ (intermediate value theorem)

- You get exact integration if the function, f, is linear ($f'' = 0$)

Example

Integrate from $f(x) = e^{-x^2}$ $a = 0$ to $b = 2$

Use trapezoidal rule:

$$
\begin{aligned}
I &= \int_0^2 e^{-x^2} dx \\
&\approx \frac{(b-a)}{2}\left[f(a) + f(b)\right] = \frac{(2-0)}{2}\left[f(2) + f(0)\right] \\
&= 1 \times (e^{-4} + e^{0}) = 1.0183
\end{aligned}
$$

Example

Estimate error: $$E_t = -\frac{1}{12} f''(\xi) h^3$$

Where $h = b - a$ and $a < \xi < b$

Don't know ξ - use average value

$$f''(x) = (-2 + 4x^2)e^{-x^2}$$

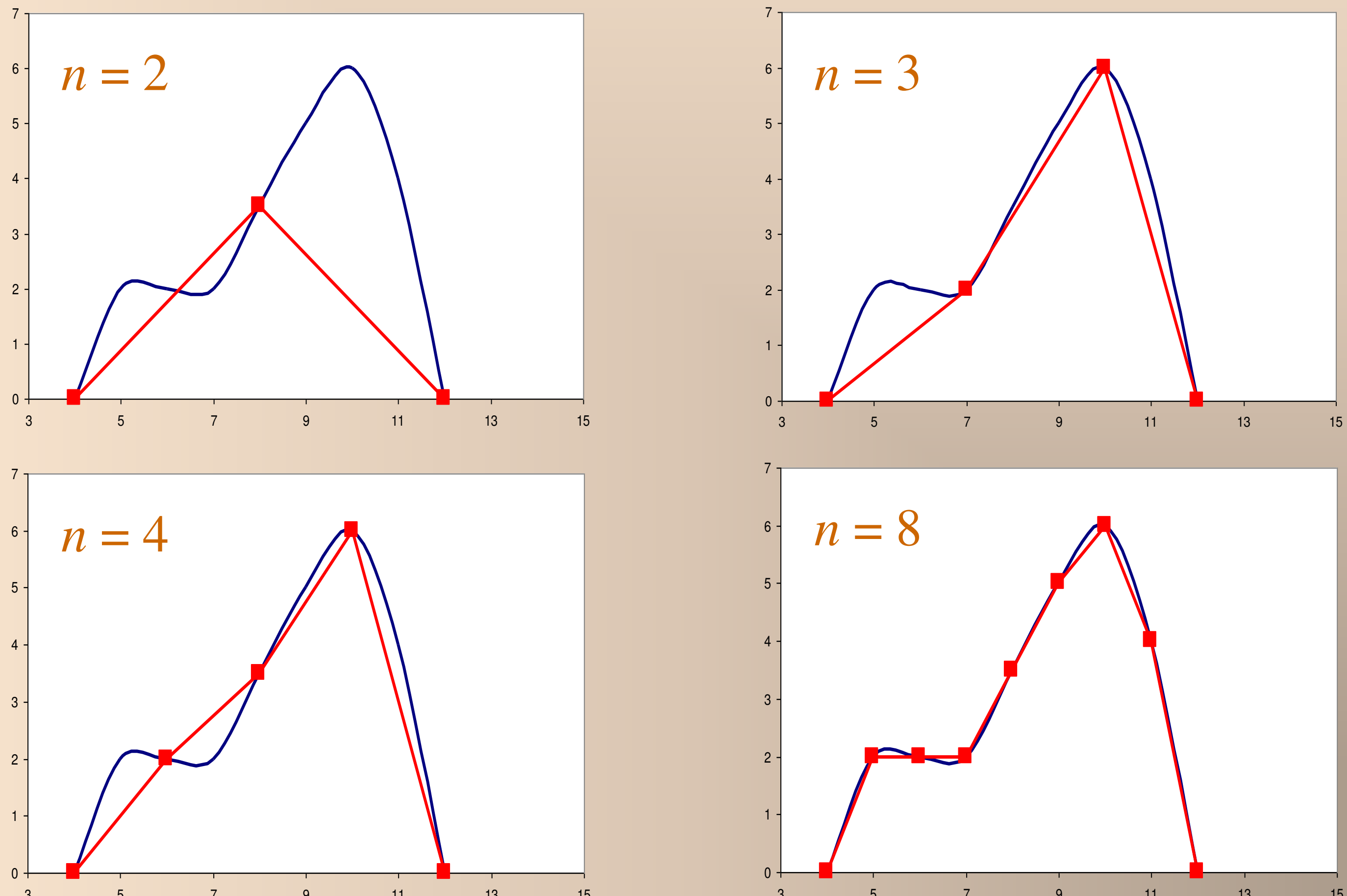
$$h = 2 - 0 = 2$$

$$f''(0) = -2$$

$$f''(2) = 0.2564$$

$$E_t \approx E_a = -\frac{2^3}{12} \frac{[f''(0) + f''(2)]}{2} = 0.58$$

More intervals, better result [error ~ $O(h^2)$]



Composite Trapezoid Rule

- If we do multiple intervals, we can avoid duplicate function evaluations and operations:
- Use n+1 <u>equally spaced points</u>.
- Each interval has: $h = \frac{b-a}{n}$
- Break up the limits of integration and expand.

$$I = \int_{a}^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{b-h}^{b} f(x)dx$$

Composite Trapezoid Rule

- Substituting the trapezoid rule for each integral.

$$I = \int_a^{a+h} f(x)\,dx + \int_{a+h}^{a+2h} f(x)\,dx + \quad \dots \quad + \int_{b-h}^{b} f(x)\,dx$$

$$= \frac{(a+h-a)}{2}\left[f(a)+f(a+h)\right] + \frac{(a+2h-a-h)}{2}\left[f(a+h)+f(a+2h)\right]$$

$$+\dots + \frac{(b-b+h)}{2}\left[f(b-h)+f(b)\right]$$

- Results in the Composite Trapezoid Formula:

$$\boxed{I = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right]}$$

Composite Trapezoid Rule

- Think of this as the *width* times the average *height*.

$$I = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b) \right]$$

$$= \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)}{2n}}_{\text{Average height}}$$

Error

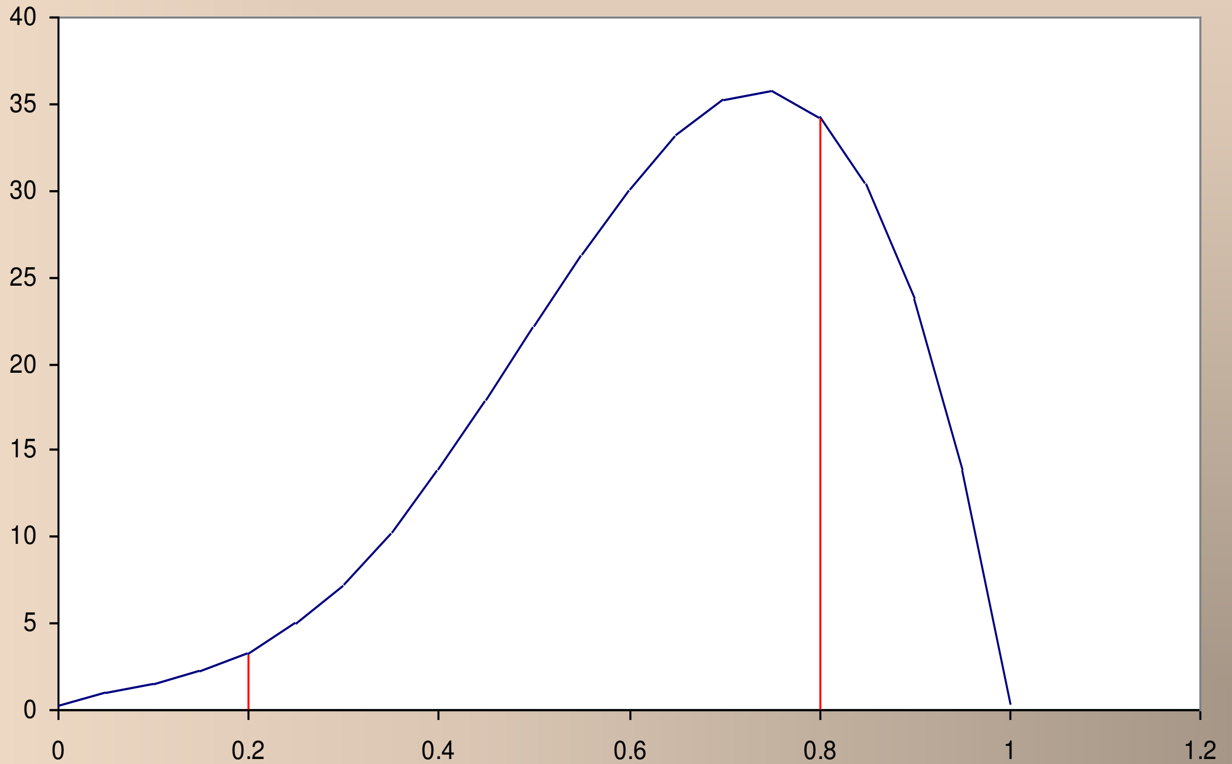
- The error can be estimated as:

$$\boxed{E_a = \frac{(b-a)h^2}{12}\bar{f}'' = \frac{(b-a)^3}{12n^2}\bar{f}''} \quad O(h^2)$$

- Where, $\bar{f}''$ is the average second derivative.
- If n is doubled, $h \rightarrow h/2$ and $E_a \rightarrow E_a/4$
- Note, that the error is dependent upon the *width* of the area being integrated.

Example

- Integrate: $f(x) = 0.3 + 20x - 140x^2 + 730x^3 - 810x^4 + 200x^5$
- from a=0.2 to b=0.8



Example

- A single application of the Trapezoid rule.

$$I = (b-a)\frac{f(a)+f(b)}{2}$$
$$= (0.8-0.2)\frac{34.22+3.81}{2}$$
$$= 11.26$$

- Error: $$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

Example

- We don't know ξ so approximate with average f''

$$f'(x) = 20 - 280x + 2190x^2 - 3240x^3 + 1000x^4$$

$$f''(x) = -280 + 4380x - 9720x^2 + 4000x^3$$

$$\bar{f}''(x) = \frac{\int_{0.2}^{0.8} f''\,dx}{0.8 - 0.2}$$

$$= \frac{f'(0.8) - f'(0.2)}{0.8 - 0.2} = -131.6$$

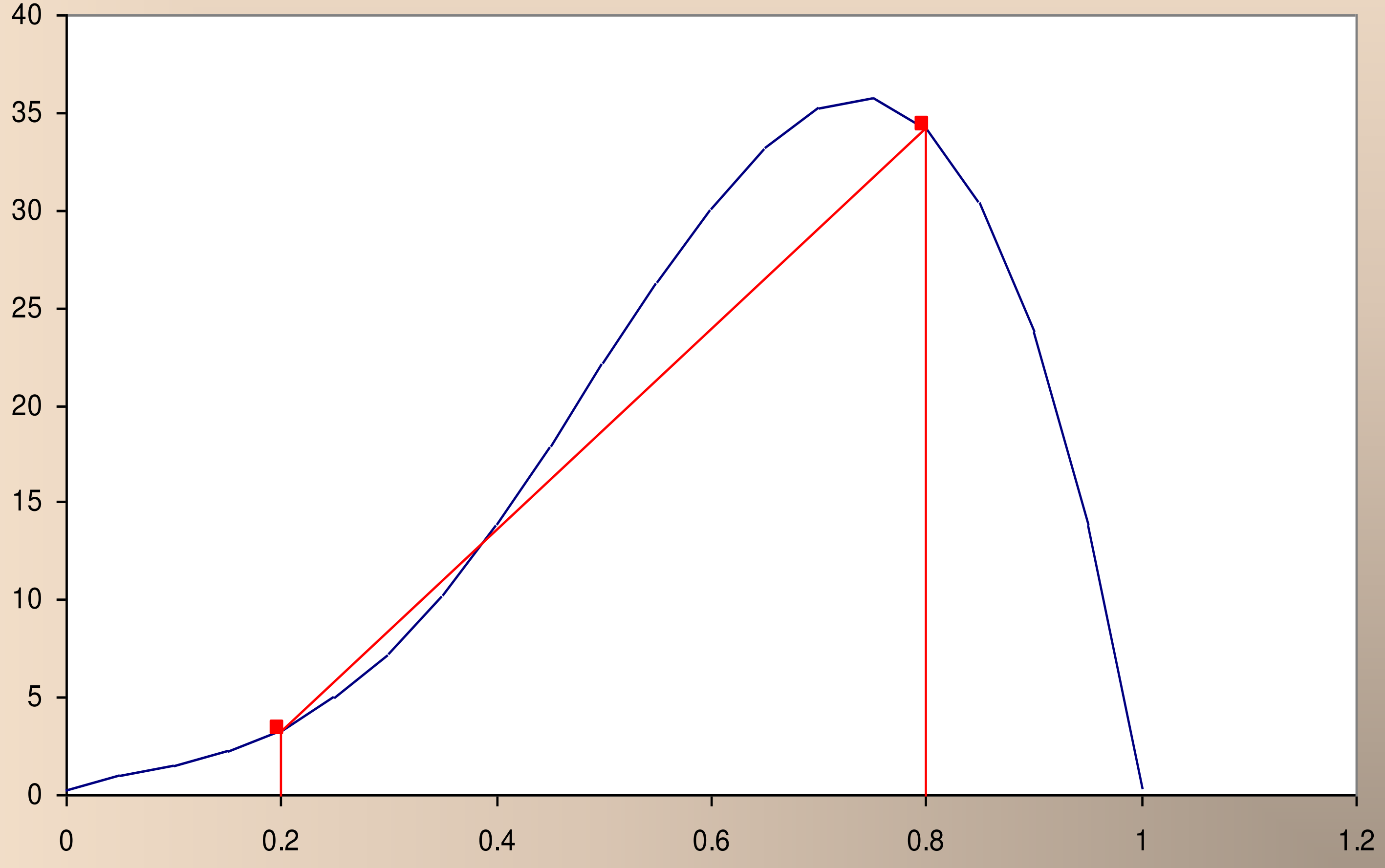
Example

- The error can thus be estimated as:

$$E_t = \frac{(b-a)h^2}{12}\bar{f}'' = \frac{(b-a)^3}{12n^2}\bar{f}''$$

$$= -\frac{1}{12}(-131.6)(0.8-0.2)^3 = 2.37$$

True value of integral is 12.82. Trapezoid rule is 11.26 - within approx error - E_t is 12%



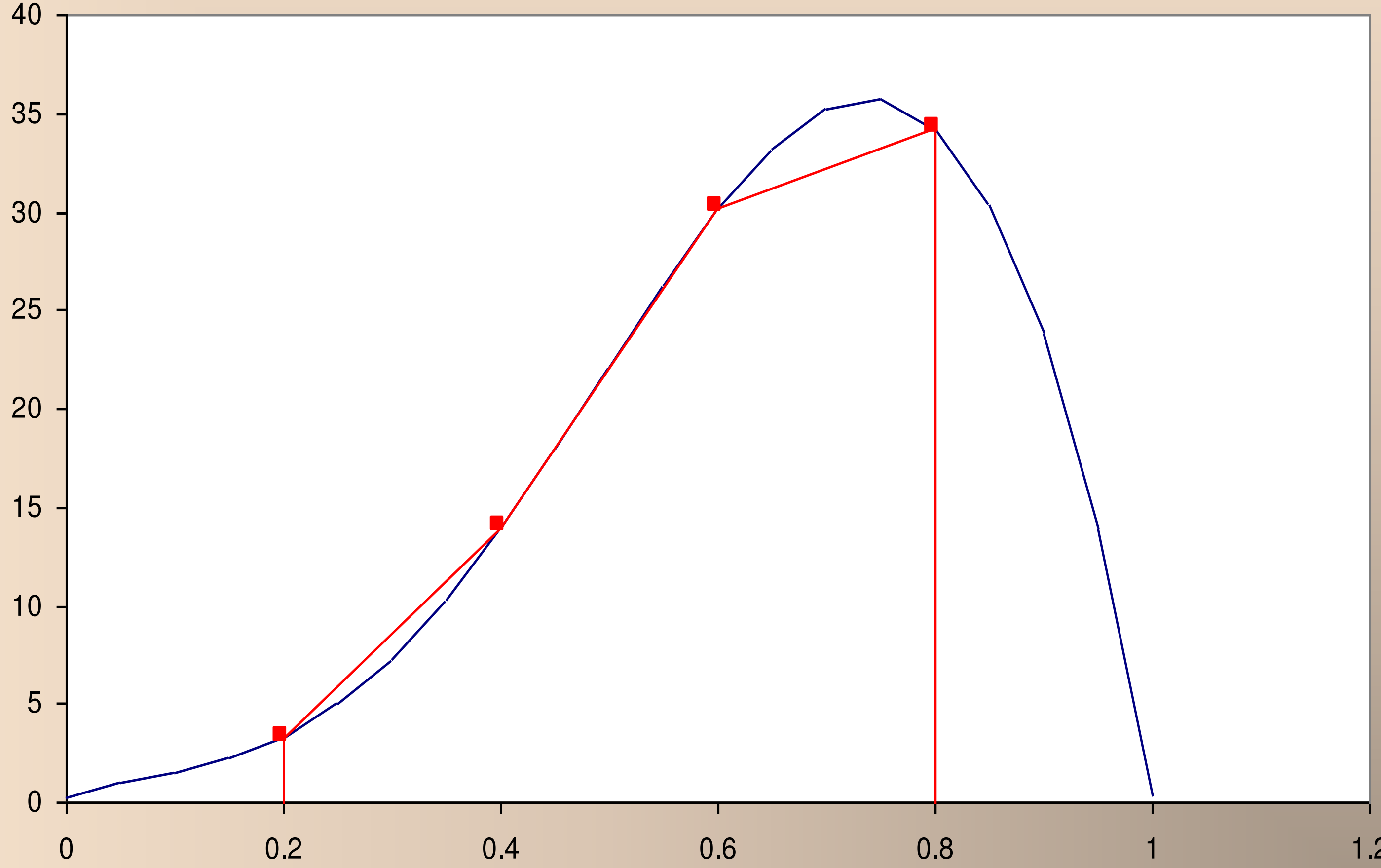
Using Three Intervals

- Use intervals (0.2,0.4),(0.4,0.6),(0.6,0.8):
  - (n = 3, h = 0.2)

$$
\begin{aligned}
I &= (b-a)\frac{f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)}{2n} \\
&= (0.8-0.2)\frac{f(0.2)+2[f(0.4)+f(0.6)]+f(0.8)}{(2)(3)} \\
&= 0.6\frac{3.31+2(13.93+30.16)+34.22}{6} \\
&= 12.57
\end{aligned}
$$

True value of integral is 12.82

E_t is now 2%



Using Six Intervals

- Use intervals (0.2,0.3),(0.3,0,4), etc.
  - $(n = 6, h = 0.1)$

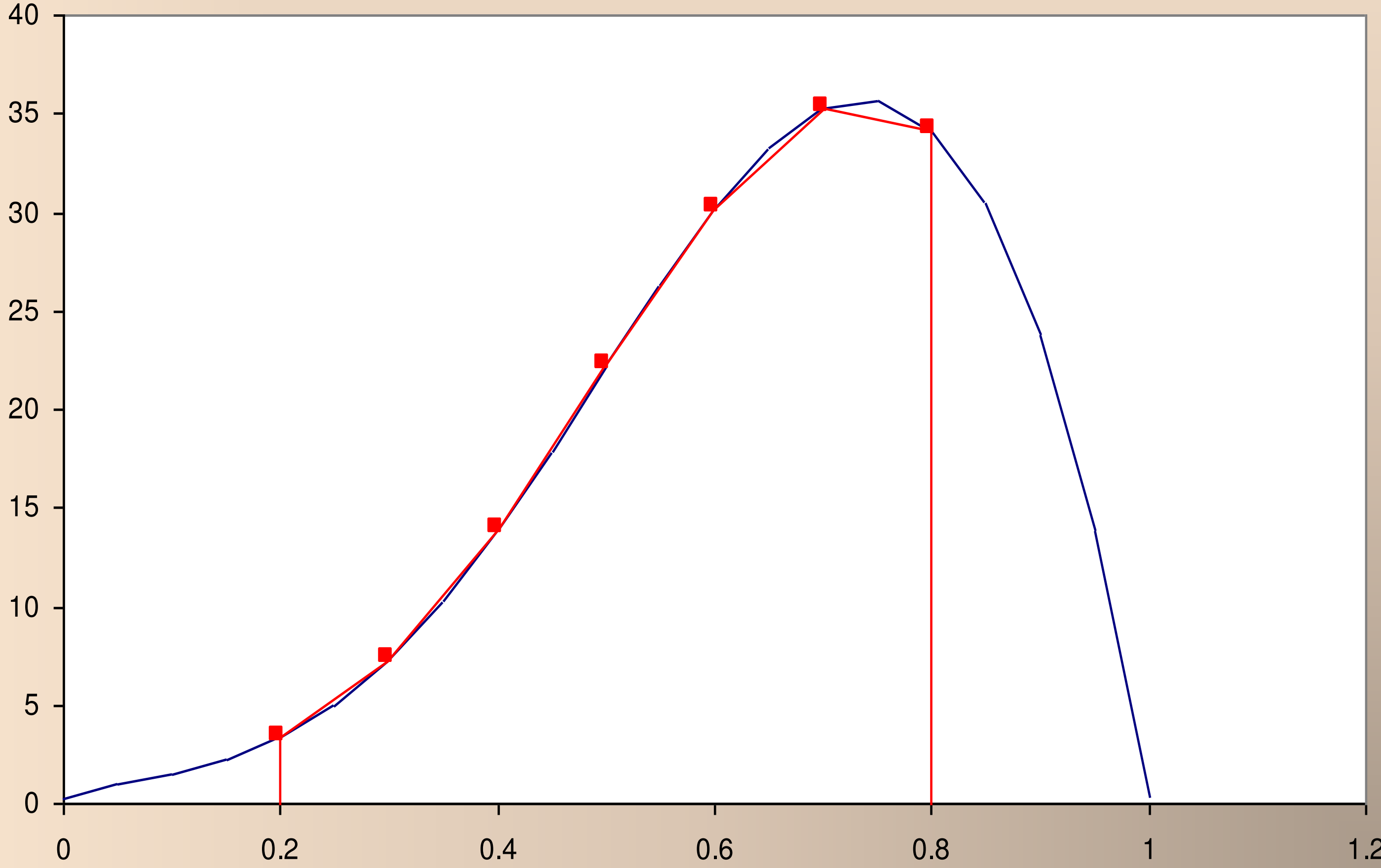
$$I = (0.8 - 0.2)\frac{f(0.2) + 2[f(0.3) + f(0.4) + f(0.5) + f(0.6) + f(0.7)] + f(0.8)}{(2)(6)}$$

$$= 0.6\frac{3.31 + 2(7.34 + 13.93 + 22.18 + 30.16 + 35.22) + 34.22}{12}$$

$$= 12.76$$

True value of integral is 12.82

E_t is now 0.5%



Example

- Trapezoid Rules:

	intervals	h	Integral ($k = 0$)	$k = 1$
$j = 0$	1	0.8	0.1728	
$j = 1$	2	0.4	1.0688	
$j = 2$	4	0.2	1.4848	

$$I = \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728) = 1.3674667$$

(j=1, k=1)

Exact integral is 1.64053334

Example

segments	h	$k=0$ $O(h^2)$	$k=1$ $O(h^4)$
1	0.8	0.1728	
2	0.4	1.0688	1.3674667
4	0.2	1.4848	

$$I=\frac{4}{3}(1.4848)-\frac{1}{3}(1.0688)=1.62346667$$

(j=2, k=1)

Exact integral is 1.64053334

Example

		$k = 1$	$k = 2$
segments	h	$O(h^2)$	$O(h^4)$
1	0.8	0.1728	
2	0.4	1.0688	1.3674667
4	0.2	1.4848	1.62346667

(j=2, k=2) $$I = \frac{16}{15}(1.62346667) - \frac{1}{15}(1.3674667) = 1.64053334$$

Exact integral is 1.64053334

Example

j \ k segments	h	$k=1$ $O(h^2)$	$k=2$ $O(h^4)$	$k=3$ $O(h^6)$
1	0.8	0.1728		
2	0.4	1.0688	1.3674667	
4	0.2	1.4848	1.62346667	1.64053334

Example

- Better and better results can be obtained by continuing this

k $\rightarrow$, j $\downarrow$ $\quad\quad k = 3$

segments	h	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
1	0.8	0.1728			
2	0.4	1.0688	1.3674667		
4	0.2	1.4848	1.62346667	1.64053334	
8	0.1	??	??	??	??

(*j*=3, *k*=3)

$$I = \frac{64}{63}(??) - \frac{1}{63}(1.64053334) = ??$$

Higher-Order Polynomials

- Recall:

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_{0+m_1}} p_{m_1}(x)dx + \int_{x_{0+m_1}}^{x_{0+m_1+m_2}} p_{m_2}(x)dx + \ldots + \int_{x_{n-m_n}}^{x_n} p_{m_n}(x)dx$$

m	Polynomial	Formula	Error
1	linear	Trapezoid	$O(h^2)$
2	quadratic	Simpson's 1/3	$O(h^4)$
3	cubic	Simpson's 3/8	$O(h^4)$