Solutions of differential equations using Laplace Transform

we can now solve differential equations using Laplace Transforms. What we do is transform the differential equation into an algebratic relation using Lapalce Transforms, solve for the Laplace Transform of the solution, then use inverse Laplace Transforms to bring the solution back into the time domain. The most important property of the Laplace transform in this case is the differentiation property:

$$
\begin{aligned}
\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} &= sF(s) - f(0^-), \\
\mathcal{L}\left\{\frac{d^2}{dt^2}f(t)\right\} &= s\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} - \frac{df}{dt}(0^-), \\
&= s^2F(s) - sf(0^-) - \frac{df}{dt}(0^-), \\
\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\} &= s^nF(s) - s^{n-1}f(0^-) - \\
&\quad \cdots - sf^{(n-2)}(0^-) - f^{(n-1)}(0^-)
\end{aligned}
$$

Here is a simple differential equation

$$\frac{dx}{dt} = 1 \quad t \geq 0$$

Since the functions are equal on the left and right, the Laplace Transforms will be equal

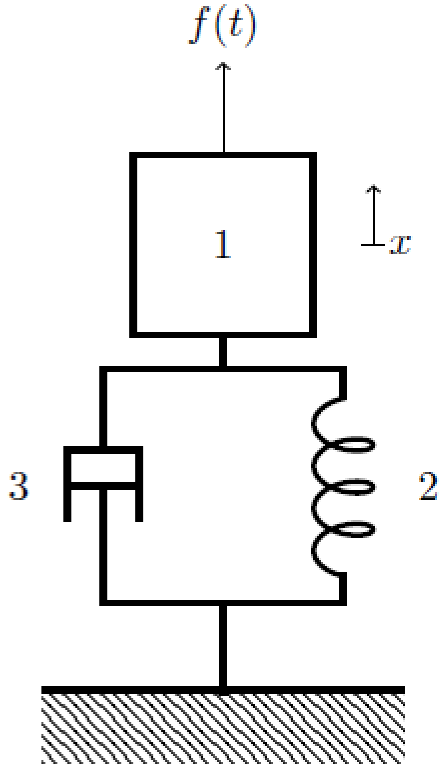
$$sX(s) - x(0^-) = \frac{1}{s}$$

$$X(s) = \frac{1}{s^2} + \frac{x(0^-)}{s}$$

$$x(t) = t + x(0^-) \quad t \geq 0$$

Mass-spring-damper System

At time $t = 0$, this system is at position $x = 0$, but with an initial velocity of 1 m/s. Beginning at this time, the force $f(t) = e^{-3t}$ is applied. Find $x(t)$ for $t \geq 0$.



We can state this problem as solving

$$\ddot{x} + 3\dot{x} + 2x = e^{-3t} \qquad x(0) = 0,\ \dot{x}(0) = 1$$

$$
\begin{aligned}
\left(s^2X(s)-1\right)+3sX(s)+2X(s) &= \frac{1}{s+3}, \\
\left(s^2+3s+2\right)X(s) &= \frac{1}{s+3}+1=\frac{s+4}{s+3}, \\
X(s) &= \frac{s+4}{(s^2+3s+2)(s+3)}, \\
&= \frac{s+4}{s^3+6s^2+11s+6}
\end{aligned}
$$

$$
X(s)=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}
$$

$$
x(t)=1.5e^{-t}-2e^{-2t}+0.5e^{-3t}
$$

Regions of convergence (ROC)

- Laplace Transform does not converge to a finite value for all signals and all values of s.
- The values of s for which Laplace transform converges is called Region of convergence (ROC).