One sided and Bilateral Z-transforms, ZT of some common signals and ROC

The z-Transform: Introduction

- Why z-Transform?
  1. Many of signals (such as $x(n)=u(n)$, $x(n) = (0.5)^n u(-n)$, $x(n) = \sin(n\omega)$ etc.) do not have a DTFT.
  2. Advantages like Fourier transform provided:
     - Solution process reduces to a simple algebraic procedures
     - The temporal domain sequence output $y(n) = x(n)*h(n)$ can be represent as $Y(z)= X(z)H(z)$
     - Properties of systems can easily be studied and characterized in z – domain (such as stability..)
- Topics:
  - Definition of z –Transform
  - Properties of z- Transform
  - Inverse z- Transform

Definition of the z-Transform

1. Definition:The z-transform of a discrete-time signal *x(n)* is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

   where $z = re^{jw}$ is a complex variable. The values of z for which the sum converges define a region in the z-plane referred to as the *region of convergence* (ROC).
2. Notationally, if *x(n)* has a z-transform *X(z),* we write

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

3. The z-transform may be viewed as the DTFT or an exponentially weighted sequence. Specifically, note that with $z = re^{jw}$, $X(z)$ *can be looked as the DTFT of the sequence* $r^{-n}x(n)$ *and ROC is determined by the range of values of r of the following right inequation.*

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [r^{-n} x(n)] e^{-jn\omega} \qquad \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

ROC & z-plane

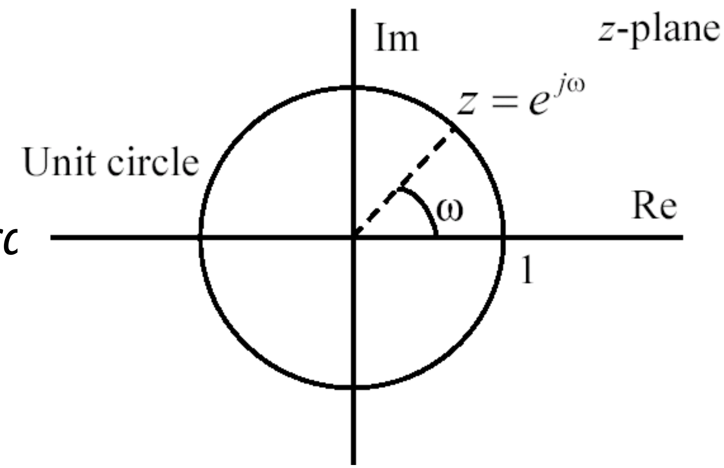
- Complex z-plane

  $z = \mathcal{R}e(z) + jIm(z) = re^{jw}$

- *Zeros and poles of X(z)*

  Many signals have z-transforms that are rc
  function of z:

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b(k)z^{-k}}{\sum_{k=0}^{p} a(k)z^{-k}}$$



  Factorizing it will give:

$$X(z) = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}$$

  The roots of the numerator polynomial, $\beta_{k,}$ are referred to as the zeros (o) and α_k are referred to as poles (x). ROC of X(z) will not contain poles.

ROC properties

- ROC is an annulus or disc in the z-plane centred at the origin. i.e. $0 \leq r_R < |z| < r_L \leq \infty$.
- A finite-length sequence has a z-transform with a region of convergence that includes the entire z-plane except, possibly, $z = 0$ and $z = \infty$. The point $z = \infty$ will be included if $x(n) = 0$ for $n < 0$, and the point $z = 0$ will be included if $x(n) = 0$ for $n > 0$.
- A right-sided sequence has a z-transform with a region of convergence that is the *exterior* of a circle:

  $$ROC: \ |z| > \alpha$$

- A left-sided sequence has a z-transform with a region of convergence that is the *interior* of a circle:

  $$ROC: \ |z| < \beta$$

- The Fourier Transform of x(n) converges absolutely if and only if ROC of z-transform includes the unit circle

Example – Right-Sided Exponential Sequence (1)

- Consider $x[n]=a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence.

$$X(z)=\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require

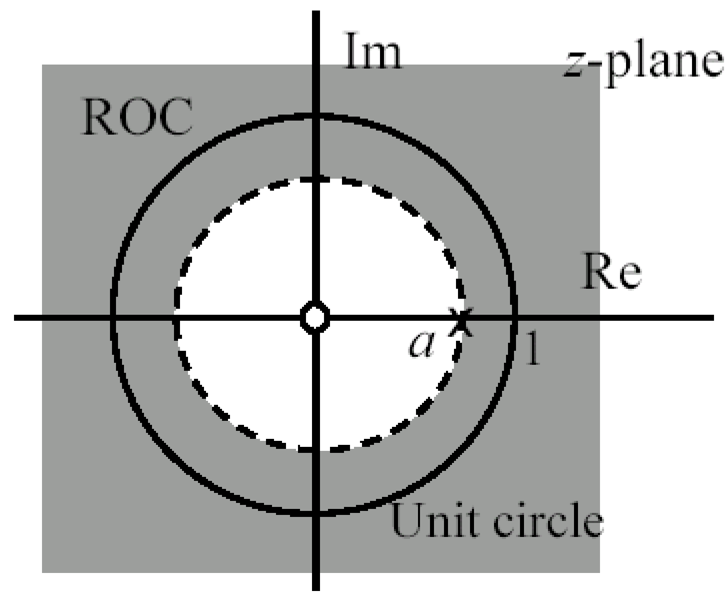
$$\sum_{n=-\infty}^{\infty} \left| az^{-1} \right|^n < \infty$$

Thus, the ROC is the range of values of z for which $|az^{-1}|<1$, or equivalently, $|z|>|a|$. Inside the ROC, the infinite series converges to

$$X(z)=\frac{1}{1-az^{-1}}=\frac{z}{z-a} \qquad |z|>|a|$$

Example – Right-Sided Exponential Sequence (2)

- The infinite sum becomes a simple rational function of z inside the ROC.
- Such a z-transform is determined to within a constant multiplier by its zeros and its poles.
- For this example,
  one zero: z=0 (plotted as o);
  one pole: z=a (plotted as x).
- When $|a|<1$, the ROC includes the unit circle.

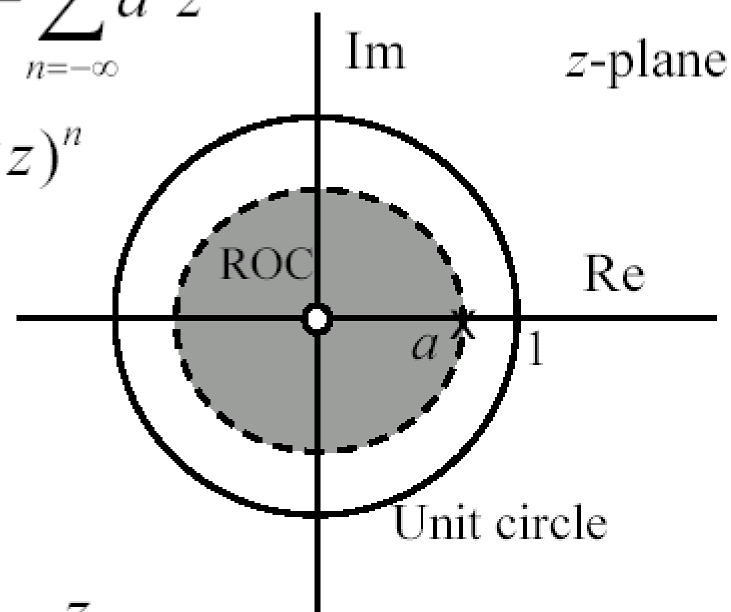


Example – Left-Sided Exponential Sequence

- Consider $x[n] = -a^n u[-n-1]$. Because it is nonzero only for $n \le -1$, this is an example of a *left-sided* sequence.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$



ROC

$$\left|a^{-1} z\right| < 1 \qquad |z| < |a|$$

and

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a} \qquad |z| < |a|$$

Notes on ROC

$$x[n] = a^n u[n] \quad \longleftrightarrow \quad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \qquad |z| > |a|$$

$$x[n] = -a^{-n} u[-n-1]$$

$$\longleftrightarrow \quad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \qquad |z| < |a|$$

- As can be seen from the two examples, *the algebraic expression or pole-zero pattern does not completely specify the z-transform of a sequence; i.e., the ROC must also be specified.*