# Properties and theorems

# Example – Two-Sided Exponential Sequence

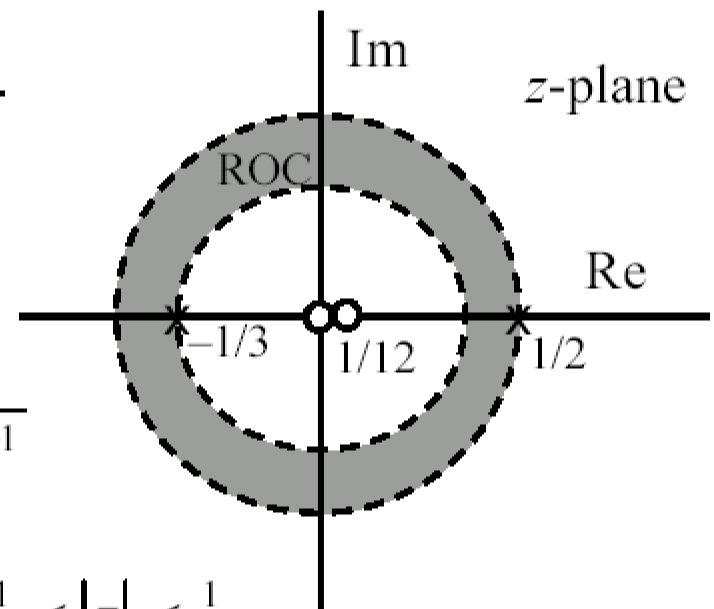
- Consider the sequence

$$x[n] = \underbrace{(-\tfrac{1}{3})^n u[n]}_{|z| > \frac{1}{3} \;\Updownarrow\; \frac{1}{1+\frac{1}{3}z^{-1}}} \; \underbrace{-(\tfrac{1}{2})^n u[-n-1]}_{|z| < \frac{1}{2} \;\Updownarrow\; \frac{1}{1-\frac{1}{2}z^{-1}}}$$

ROC $\quad \frac{1}{3} < |z| < \frac{1}{2}$

and $\quad X(z) = \dfrac{1}{1+\frac{1}{3}z^{-1}} + \dfrac{1}{1-\frac{1}{2}z^{-1}}$

$$= \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})} \qquad \frac{1}{3} < |z| < \frac{1}{2}$$



# Example – Finite-Length Sequence

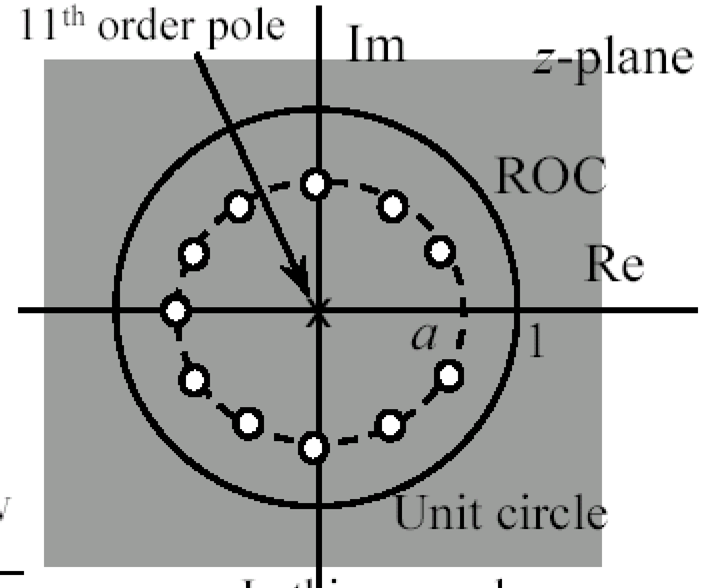
- Consider the sequence

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$




In this example $N$ = 12 and 0< $a$ <1.

The ROC is determined by $\sum_{n=0}^{N-1} | az^{-1} |^n$

which requires $|a| < \infty$ and $z \neq 0$

Table 4-1 Common $z$-Transform Pairs

| Sequence | $z$-Transform | Region of Convergence |
|---|---|---|
| $\delta(n)$ | $1$ | all $z$ |
| $\alpha^n u(n)$ | $\dfrac{1}{1-\alpha z^{-1}}$ | $\|z\| > \|\alpha\|$ |
| $-\alpha^n u(-n-1)$ | $\dfrac{1}{1-\alpha z^{-1}}$ | $\|z\| < \|\alpha\|$ |
| $n\alpha^n u(n)$ | $\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\|z\| > \|\alpha\|$ |
| $-n\alpha^n u(-n-1)$ | $\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\|z\| < \|\alpha\|$ |
| $\cos(n\omega_0)u(n)$ | $\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| $\sin(n\omega_0)u(n)$ | $\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |

# Properties of Z-Transform

- Linearity

  If x(n) has a z-transform X(z) with a region of convergence Rx, and if y(n) has a z-transform Y(z) with a region of convergence Ry,

$$w(n) = ax(n) + by(n) \xleftrightarrow{Z} W(z) = aX(z) + bY(z)$$

  and the ROC of W(z) will include the intersection of Rx and Ry, that is, Rw contains .

- Shifting property $R_x \bigcap R_y$

  If x(n) has a z-transform X(z),

- Time reversal

$$x(n - n_0) \xleftrightarrow{Z} z^{-n_0} X(z)$$

  If x(n) has a z-transform X(z) with a region of convergence Rx that is the annulus , the z-transform of the time-reversed sequence x(-n) is $\alpha < |z| < \beta$

  and has a region of convergence $x(-n) \xleftrightarrow{Z} X(z^{-1})$, which is denoted by

$1/\beta < |z| < 1/\alpha$ $1/R_x$

# Properties of Z-Transform

- Multiplication by an exponential
  - If a sequence x(n) is multiplied by a complex exponential $\alpha^n$.

$$\alpha^n x(n) \xleftrightarrow{Z} X(\alpha^{-1} z)$$

- Convolution theorm

  If x(n) has a z-transform X(z) with a region of convergence $R_x$, and if h(n) has a z-transform H(z) with a region of convergence $R_h$,

$$y(n) = x(n) * h(n) \xleftrightarrow{Z} Y(z) = X(z)H(z)$$

  The ROC of Y(z) will include the intersection of $R_x$ and $R_h$, that is,

  $R_y$ contains $R_x \cap R_h$ .

  With x(n), y(n), and h(n) denoting the input, output, and unit-sample response, respectively, and X(z), Y(x), and H(z) their z-transforms. The z-transform of the unit-sample response is often referred to as the system function.

- Conjugation

  If X(z) is the z-transform of x(n), the z-transform of the complex conjugate of x(n) is

$$x^*(n) \xleftrightarrow{Z} X^*(z^*)$$

# Properties of Z-Transform

- Derivative
  - If X(z) is the z-transform of x(n), the z-transform of  is

$$nx(n) \xleftrightarrow{z} -z\frac{dX(z)}{dz}$$

- Initial value theorem

  If X(z) is the z-transform of x(n) and x(n) is equal to zero for n<0, the initial value, x(0), maybe be found from X(z) as follows:

$$x(0) = \lim_{z \to \infty} X(z)$$

**Table 4-2 Properties of the $z$-Transform**

| Property | Sequence | $z$-Transform | Region of Convergence |
|---|---|---|---|
| Linearity | $ax(n) + by(n)$ | $aX(z) + bY(z)$ | Contains $R_x \cap R_y$ |
| Shift | $x(n - n_0)$ | $z^{-n_0}X(z)$ | $R_x$ |
| Time reversal | $x(-n)$ | $X(z^{-1})$ | $1/R_x$ |
| Exponentiation | $\alpha^n x(n)$ | $X(\alpha^{-1}z)$ | $\|\alpha\|R_x$ |
| Convolution | $x(n) * y(n)$ | $X(z)Y(z)$ | Contains $R_x \cap R_y$ |
| Conjugation | $x^*(n)$ | $X^*(z^*)$ | $R_x$ |
| Derivative | $nx(n)$ | $-z\dfrac{dX(z)}{dz}$ | $R_x$ |

*Note:* Given the $z$-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence $R_x$ and $R_y$, respectively, this table lists the $z$-transforms of sequences that are formed from $x(n)$ and $y(n)$.