Solution of difference equations using one-sided ZT

Properties of z-Transform

(1) Time Shifting $\quad x[n-n_0] \longleftrightarrow z^{-n_0}X(z),$

The rationality of $X(z)$ unchanged, ROC unchanged except for the possible addition or deletion of the origin or infinity

$n_o > 0 \Rightarrow$ ROC $z \neq 0$ (maybe)

$n_o < 0 \Rightarrow$ ROC $z \neq \infty$ (maybe)

(2) z-Domain Differentiation $\quad nx[n] \longleftrightarrow -z\dfrac{dX(z)}{dz},$

same ROC

(3) Linearity : $\quad ax[n]+by[n] \longleftrightarrow aX(z)+bY(z)$

Properties of z-Transform

(4) Z-scale Property: $a^n x[n] \longleftrightarrow X\left(\dfrac{z}{a}\right)$

(5) Initial Value : $x[0] = \lim_{z \to \infty} X(z)$

(6) Final Value : $x[\infty] = \lim_{z \to 1} (z-1) X(z)$

(Applicable only if the ROC of $(z-1)X(z)$ includes the unit circle, i.e., all the poles are inside the unit circle)

(7) Convolution : $h[n] * x[n] \longleftrightarrow H(z) X(z)$

Rational z-Transform

For most practical signals, the z-transform can be expressed as a ratio of two polynomials

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0(z - z_1)(z - z_2)\cdots(z - z_M)}{a_0(z - p_1)(z - p_2)\cdots(z - p_N)}$$

where $z_1, z_2, \cdots, z_M$ are the *zeroes* of $X(z)$, i.e., the roots of the numerator polynomial

and $p_1, p_2, \cdots, p_N$ are the *poles* of $X(z)$, i.e., the roots of the denominator polynomial.

Rational z-Transform

It is customary to normalize the denominator polynomial to make its leading coefficients one, i.e.,

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

$$= \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Also, it $x[n]$ is a causal signal, then $X(z)$ will be a proper rational polynomial with $M \le N$, i.e., # of zeroes $\le$ # of poles.

Inverse z-Transform

$$X(z) = X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\}, \text{ where } z = re^{j\omega} \in \text{ ROC}$$

DTFT

$$x[n]r^{-n} = \mathcal{F}^{-1}\left\{X(re^{j\omega})\right\} = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

IDTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\underbrace{r^n e^{j\omega n}}_{z^n}d\omega$$

$$= \frac{1}{2\pi j}\oint X(z)z^{n-1}dz \quad \text{(A contour integral)}$$

where, for a fixed r, $z = re^{j\omega} \Rightarrow dz = jre^{j\omega}d\omega \Rightarrow d\omega = \frac{1}{j}z^{-1}dz$

Synthetic Division Method

- Write $X(z)$ as a normalized rational polynomial in z^{-1} by multiplying the numerator and denominator by z^{-N}

$$X(z) = \frac{z^{-r}(b_0 + b_1 z^{-1} + \cdots + b_M z^{-M})}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

- Perform long division of the numerator polynomial by the denominator polynomial to produce the quotient polynomial $q(z^{-1})$

- Identify coefficients in the power series definition of $X(z)$ where

$$X(z) = z^{-r}[q(0) + q(1)z^{-1} + q(2)z^{-2} + \cdots]$$

$$r = 0 \rightarrow x[n] = q[n], \qquad r \geq 0 \rightarrow x[n] = \begin{cases} 0, & 0 \leq n \leq r \\ q[n-r], & r \leq n < \infty \end{cases}$$

Ex. Find the inverse z-transform of $X(z) = 3z^{3} - z + 2z^{-4}$

$$X(z) = 3z^{-(-3)} - z^{-(-1)} + 2z^{-4}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots x[-3]z^{-(-3)} + x[-2]z^{-(-2)} + x[-1]z^{-(-1)}$$

$$+ x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

Equating coefficients,

$$x[n] = \{\cdots, 0, 3, 0, -1, \underset{\uparrow}{0}, 0, 0, 0, 2, 0, \cdots\}$$

Remarks: This method doesn't produce a closed-form expression for $x[n]$

Z-Transform Solution of Linear Difference Equations

- **We can use z-transform to solve the difference equation that characterizes a causal, linear, time invariant system. The following expressions are especially useful to solve the difference equations:**
- $z[y[(n-1)T] = z^{-1}Y(z) + y[-T]$
- $Z[y(n-2)T] = z^{-2}Y(z) + z^{-1}y[-T] + y[-2T]$
- $Z[y(n-3)T] = z^{-3}Y(z) + z^{-2}y[-T] + z^{-1}y[-2T] +$
  $y[-3T]$

Example: Consider the following difference equation: $y[nT] - 0.1y[(n-1)T] - 0.02y[(n-2)T] = 2x[nT] - x[(n-1)T]$ where the initial conditions are $y[-T] = -10$ and $y[-2T] = 20$. $Y[nT]$ is the output and $x[nT]$ is the unit step input.

Solution:

Computing the z-transform of the difference equation gives

$$Y(z) - 0.1[z^{-1}Y(z) + y[-T]] - 0.02[z^{-2}Y(z) + z^{-1}y[-T] + y[-2T]] = 2X(z) - z^{-1}X(z)$$

Substituting the initial conditions we get

$$Y(z) - 0.1z^{-1}Y(z) + 1 - 0.02z^{-2}Y(z) - 0.2z^{-1} - 0.4 = (2 - z^{-1})X(z)$$

$$\left(1-0.1z^{-1}-0.02z^{-2}\right)Y(z)=\left(2-z^{-1}\right)\frac{1}{1-z^{-1}}-0.2z^{-1}-0.6$$

$$Y(z)\left[1-0.2z^{-1}-0.02z^{-2}\right]=\frac{2-z^{-1}}{1-z^{-1}}-0.2z^{-1}-0.6$$

$$Y(z)=\frac{1.4-0.6z^{-1}+0.2z^{-2}}{\left(1-z^{-1}\right)\left(1-0.1z^{-1}-0.02z^{-2}\right)}=\frac{1.4-0.6z^{-1}+0.2z^{-2}}{\left(1-z^{-1}\right)\left(1-0.2z^{-1}\right)\left(1+0.1z^{-1}\right)}$$

$$=\frac{1.4z^{3}-0.6z^{2}+0.2z}{(z-1)(z-0.2)(z+0.1)}$$

$$\frac{Y(z)}{z}=\frac{1.136}{z-1}+\frac{-0.567}{z-0.2}+\frac{0.830}{z+0.1}$$

$$Y(z)=1.136\frac{1}{1-z^{-1}}-0.567\frac{1}{1-0.2z^{-1}}+0.830\frac{1}{1+0.1z^{-1}}$$

and the output signal y[nT] is

$$y[nT]=1.136u[nT]-0.567(0.2)^{n}u[nT]+0.830(-0.1)^{n}u[nT]$$