Some important FT theorems, Parseval's theorem

Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

Parseval Theorem

Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$$

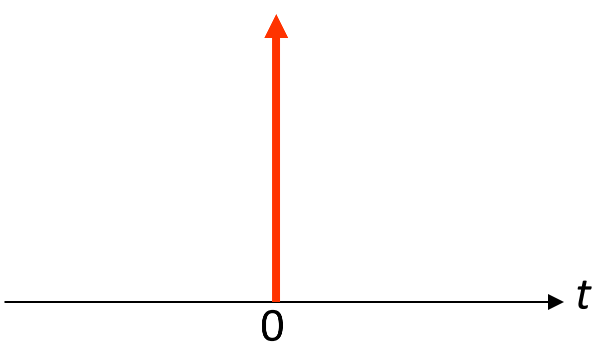
$$\int_{-\infty}^{\infty} f(x) g^*(x) \; dx = \int_{-\infty}^{\infty} F(\xi) G^*(\xi) \; d\xi$$

$f(x)$	$F(\xi)$
Real(R)	Real part even (RE) Imaginary part odd (IO)
Imaginary (I)	RO,IE
RE,IO	R
RE,IE	I
RE	RE
RO	IO
IE	IE
IO	RO
Complex even (CE)	CE
CO	CO

Fourier Transform Theorem

Dirac Delta Function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$ and $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



Also called *unit impulse function*.

Generalized Function

- The value of delta function can also be defined in the sense of generalized function:

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)}$$

$\phi(t)$: Test Function

- We shall never talk about the value of $\delta(t)$.
- Instead, we talk about the values of integrals involving $\delta(t)$.

Properties of Unit Impulse Function

$$\boxed{\int_{-\infty}^{\infty} \delta(t - t_0)\phi(t)\,dt = \phi(t_0)}$$

Pf)

Write t as $t + t_0$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\phi(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\phi(t + t_0)\,dt$$

$$= \phi(t_0)$$

Properties of Unit Impulse Function

$$\boxed{\int_{-\infty}^{\infty} \delta(at)\phi(t)dt = \frac{1}{|a|}\phi(0)}$$

Pf) Write *t* as *t*/*a*

Consider $a>0$

$$\int_{-\infty}^{\infty} \delta(at)\phi(t)dt$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} \delta(t)\phi\left(\frac{t}{a}\right)dt$$

$$= \frac{1}{|a|}\phi(0)$$

Consider $a<0$

$$\int_{-\infty}^{\infty} \delta(at)\phi(t)dt$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} \delta(t)\phi\left(\frac{t}{a}\right)dt$$

$$= \frac{1}{|a|}\phi(0)$$

Properties of Unit Impulse Function

$$\boxed{f(t)\delta(t) = f(0)\delta(t)}$$

Pf)

$$\begin{aligned}
\int_{-\infty}^{\infty} [\underbrace{f(t)\delta(t)}]\phi(t)\,dt &= \int_{-\infty}^{\infty} \delta(t)[f(t)\phi(t)]\,dt \\
&= f(0)\phi(0) \\
&= f(0)\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt \\
&= \int_{-\infty}^{\infty} [\underbrace{f(0)\delta(t)}]\phi(t)\,dt
\end{aligned}$$

Properties of Unit Impulse Function

$$\boxed{f(t)\delta(t) = f(0)\delta(t)}$$

$$\boxed{\delta(at) = \frac{1}{|a|}\delta(t)}$$

Pf)

$$\int_{-\infty}^{\infty} \underbrace{\delta(at)}\phi(t)\,dt = \frac{1}{|a|}\phi(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{|a|}\delta(t)}\phi(t)\,dt$$

Properties of Unit Impulse Function

$$f(t)\delta(t) = f(0)\delta(t)$$

$$t\delta(t) = 0$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta(-t) = \delta(t)$$

Generalized Derivatives

The derivative $f'(t)$ of an arbitrary generalized function $f(t)$ is defined by:

$$\boxed{\int_{-\infty}^{\infty} f'(t)\phi(t)dt = -\int_{-\infty}^{\infty} f(t)\phi'(t)dt}$$

Show that this definition is consistent to the ordinary definition for the first derivative of a continuous function.

$$\int_{-\infty}^{\infty} f'(t)\phi(t)dt = \underbrace{f(t)\phi(t)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f(t)\phi'(t)dt$$