Fourier Transform Theorems Contd….

Derivatives of the δ-Function

$$\int_{-\infty}^{\infty} \delta'(t)\phi(t)dt = -\int_{-\infty}^{\infty} \delta(t)\phi'(t)dt = -\phi'(0)$$

$$\delta'(t) = \frac{d\delta(t)}{dt}, \qquad \phi'(0) = \left.\frac{d\phi(t)}{dt}\right|_{t=0}$$

$$\int_{-\infty}^{\infty} \delta^{(n)}(t)\phi(t)dt = (-1)^n \phi^{(n)}(0)$$

$$\delta^{(n)}(t) = \frac{d^n\delta(t)}{dt^n}, \qquad \phi^{(n)}(0) = \left.\frac{d^n\phi(t)}{dt^n}\right|_{t=0}$$

Product Rule

$$[f(t)\delta(t)]' = f'(t)\delta(t) + f(t)\delta'(t)$$

Pf)

$$\int_{-\infty}^{\infty} \underbrace{[f(t)\delta(t)]'}\phi(t)dt = -\int_{-\infty}^{\infty} [f(t)\delta(t)]\phi'(t)dt = -\int_{-\infty}^{\infty} \delta(t)[f(t)\phi'(t)]dt$$

$$= -\int_{-\infty}^{\infty} \delta(t)\{[f(t)\phi(t)]' - f'(t)\phi(t)\}dt$$

$$= -\int_{-\infty}^{\infty} \delta(t)[f(t)\phi(t)]'dt + \int_{-\infty}^{\infty} \delta(t)[f(t)'\phi(t)]dt$$

$$= \int_{-\infty}^{\infty} \delta'(t)[f(t)\phi(t)]dt + \int_{-\infty}^{\infty} \delta(t)[f(t)'\phi(t)]dt$$

$$= \int_{-\infty}^{\infty} \underbrace{[\delta'(t)f(t) + \delta(t)f'(t)]}\phi(t)dt$$

Product Rule

$$f(t)\delta'(t) = f(0)\delta'(t) - f'(0)\delta(t)$$

Pf)

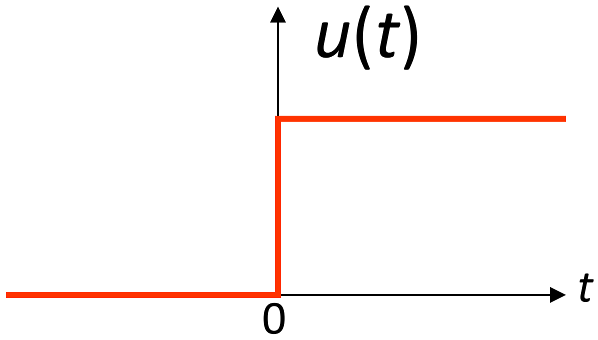
$$f(t)\delta'(t) = \underbrace{[f(t)\delta(t)]'}_{} - \underbrace{f(t)'\delta(t)}_{}$$

$$= [f(0)\delta(t)]' \qquad = f'(0)\delta(t)$$

$$= f(0)\delta'(t)$$

Unit Step Function $u(t)$

- Define $$\int_{-\infty}^{\infty} u(t)\phi(t)dt = \int_{0}^{\infty} \phi(t)dt$$



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Derivative of the Unit Step Function

- Show that $u'(t) = \delta(t)$

$$\int_{-\infty}^{\infty} \underbrace{u'(t)}\phi(t)dt = -\int_{-\infty}^{\infty} u(t)\phi'(t)dt$$

$$= -\int_{0}^{\infty} \phi'(t)dt$$

$$= -[\phi(\infty) - \phi(0)] = \phi(0)$$

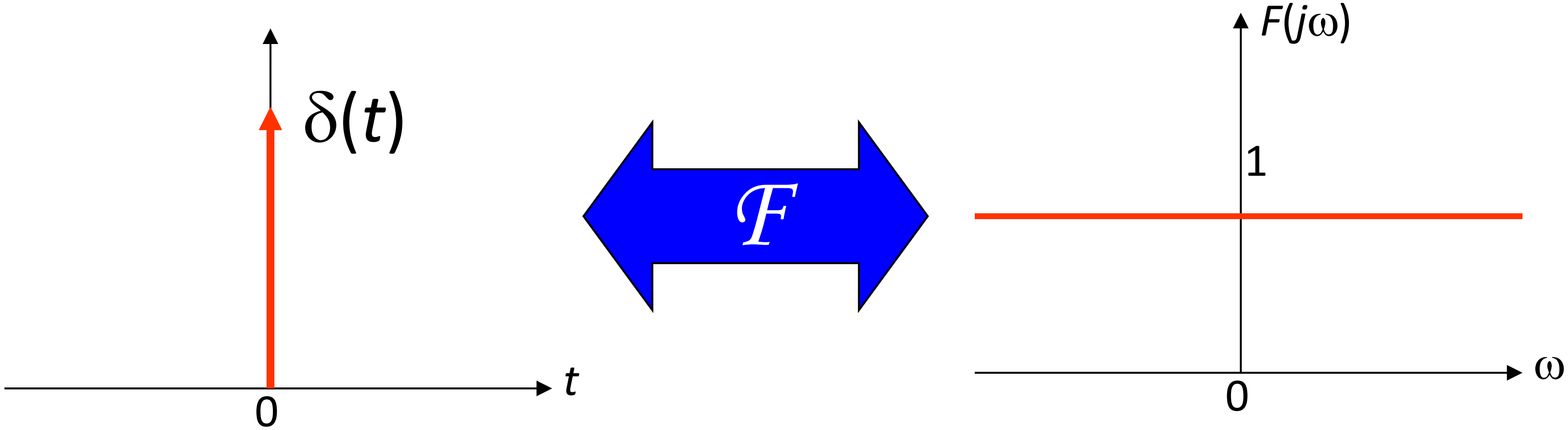
$$= \int_{-\infty}^{\infty} \underbrace{\delta(t)}\phi(t)dt$$

Derivative of the Unit Step Function

Fourier Transform for $\delta(t)$

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \;= e^{-j\omega t}\Big|_{t=0} = 1$$



Fourier Transform for $\delta(t)$

Show that
$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

$$\delta(t) = \mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

The integration $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$ converges to $\delta(t)$

in the sense of generalized function.

Fourier Transform for $\delta(t)$

Show that $$\delta(t) = \frac{1}{\pi}\int_0^{\infty} \cos\omega t\, d\omega$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\cos\omega t + j\sin\omega t) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\omega t\, d\omega + \frac{j}{2\pi}\int_{-\infty}^{\infty} \sin\omega t\, d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty} \cos\omega t\, d\omega$$

Converges to $\delta(t)$ in the sense of generalized function.

Two Identities for $\delta(t)$

$$\delta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jxy}\, dx$$

$$\delta(y) = \frac{1}{\pi}\int_{0}^{\infty} \cos xy\, dx$$

These two ordinary integrations themselves are meaningless.

They converge to $\delta(t)$ in the sense of generalized function.