Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T (i.e. a finite sequence of data).

Let $f(t)$ be the continuous signal which is the source of the data. Let N samples be denoted $f[0], f[1], f[2], \ldots, f[k], \ldots, f[N-1]$.

The Fourier Transform of the original signal, $f(t)$, would be

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

We could regard each sample $f[k]$ as an *impulse* having area $f[k]$. Then, since the integrand exists only at the sample points:

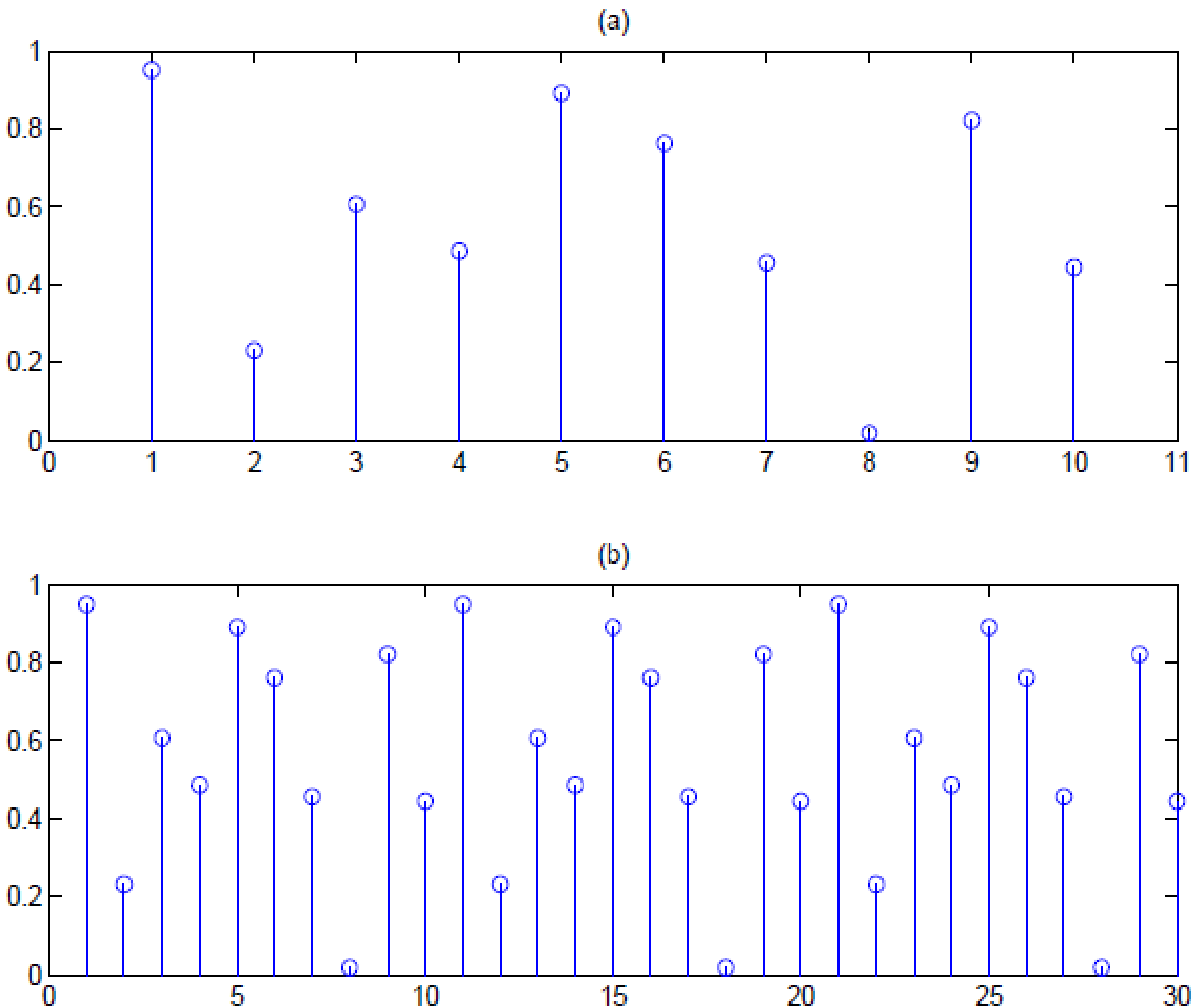
$$
\begin{aligned}
F(j\omega) &= \int_0^{(N-1)T} f(t)e^{-j\omega t}dt \\
&= f[0]e^{-j0} + f[1]e^{-j\omega T} + \ldots + f[k]e^{-j\omega kT} + \ldots f(N-1)e^{-j\omega(N-1)T}
\end{aligned}
$$

ie. $$F(j\omega) = \sum_{k=0}^{N-1} f[k]e^{-j\omega kT}$$

We could in principle evaluate this for any ω, but with only N data points to start with, only N final outputs will be significant.

You may remember that the continuous Fourier transform could be evaluated over a finite interval (usually the fundamental period T_0) rather than from $-\infty$ to $+\infty$ if the waveform was *periodic*. Similarly, since there are only a finite number of input data points, the DFT treats the data as if it were periodic (i.e. $f(N)$ to $f(2N-1)$ is the same as $f(0)$ to $f(N-1)$.)

Hence the sequence shown below in Fig. is considered to be one period of the periodic sequence in plot,


(a)
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10 11
(b)
1
0.8
0.6
0.4
0.2
0
0 5 10 15 20 25 30

Since the operation treats the data as if it were periodic, we evaluate the DFT equation for the fundamental frequency (one cycle per sequence, $\frac{1}{NT}$Hz, $\frac{2\pi}{NT}$ rad/sec.) and its harmonics (not forgetting the d.c. component (or average) at $\omega = 0$).

$$\text{i.e.} \quad \text{set } \omega = 0, \frac{2\pi}{NT}, \frac{2\pi}{NT} \times 2, \ldots \frac{2\pi}{NT} \times n, \ldots \frac{2\pi}{NT} \times (N-1)$$

or, in general

$$F[n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk} \quad (n = 0 : N-1)$$

$F[n]$ is the Discrete Fourier Transform of the sequence $f[k]$.

We may write this equation in matrix form as:

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{N-2} \\ 1 & W^3 & W^6 & W^9 & \dots & W^{N-3} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \dots & W \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

where $W = \exp(-j2\pi/N)$ and $W = W^{2N}$ etc. $= 1$.