# LTI Systems

- LTI Systems are *completely characterized* by its unit sample response
- The output of any LTI System is a convolution of the input signal with the unit-impulse response, *i.e.* y[n] = x[n]\*h[n]

$$=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$$

## **Properties of Convolution**

#### **Commutative Property**

x[n] \* h[n] = h[n] \* x[n]

#### **Distributive Property**

 $x[n]^*(h_1[n] + h_2[n]) =$ (x[n]^\*h\_1[n]) + (x[n]^\*h\_2[n])

#### **Associative Property**

 $x[n] * h_1[n] * h_2[n] =$ (x[n] \* h\_1[n]) \* h\_2[n] = (x[n] \* h\_2[n]) \* h\_1[n]

$$x[n] \xrightarrow{y[n]} = \frac{h[n]}{x[n]} \xrightarrow{y[n]}$$







 $x[n] \longrightarrow h_2[n] \longrightarrow h_1[n] \longrightarrow y[n]$ 

## **Useful Properties of (DT) LTI Systems**

• Causality:

$$h[n] = 0 \quad n < 0$$
$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

• Stability:

#### Bounded Input ↔ Bounded Output

for 
$$|x[n]| \le x_{\max} < \infty$$
  
 $|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| \le x_{\max} \left|\sum_{k=-\infty}^{\infty} h[n-k]\right| < \infty$ 

### The unit sample and unit step

Let's examine some special signals, first in discrete time, then in continuous time.

**Definition 1.1.** The discrete time unit step is given by

$$u[n] = \begin{cases} 1, n \ge 0\\ 0, n < 0 \end{cases}$$

The unit sample or impulse is defined as

$$\delta[n] = u[n] - u[n-1]$$

We notice that they are related via the sum relation

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

**Proposition 1.1.** The unit sample has the "sampling property," picking off values of signals that it sums against:

$$x[n] = \sum_{k} x[n-k]\delta[k]$$

This is true for all signals, implying we can derive various properties, the "summed" and "differenced" versions. Defining x[n] = u[n], then

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Now let's examine linearity and time invariance.

**Example 1.1.** Define the system with input x[n] and output y[n]=nx[n]. This system is linear but not time invariant.

To see linearity is straightforward, Take linear combinations of inputs and verify outputs are linear combinations. To see the system is not time invariant, define input  $x_1[n] = \delta[n]$ , then output is  $y_1[n] = n\delta[n] = 0$  for all n. Now shift the input,  $x_2[n] = \delta[n-1]$ . But for this  $y_2[1] = 1$ .

**Definition 1.2.** The continuous time version has a similar form:

$$u(t) = \begin{cases} 1, t \ge 0\\ 0, t < 0 \end{cases}$$

$$\delta(t) = \dot{u}(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (u(t+\epsilon) - u(t-\epsilon))$$

**Proposition 1.2.** *The sifting property follows for smooth functions:* 

$$x(t) = \int x(t-\tau)\delta(\tau)d\tau$$

To see this, we can argue loosely swapping the order of the limit according to

$$\begin{aligned} x(t) &= \lim_{n \to 0} \frac{1}{\epsilon} \int x(t-\tau) \big( u(\tau+\epsilon) - u(\tau-\epsilon) \big) d\tau \\ &= \lim_{n \to 0} \frac{1}{\epsilon} x(t-\tau) d\tau = x(t) \end{aligned}$$

In particular, defining x(t)=u(t), then we see the integrated property follows

$$u(t) = \int_{\sigma = -\infty}^{t} \delta(\sigma) d\sigma$$

$$\delta(t) = \dot{u}(t)$$