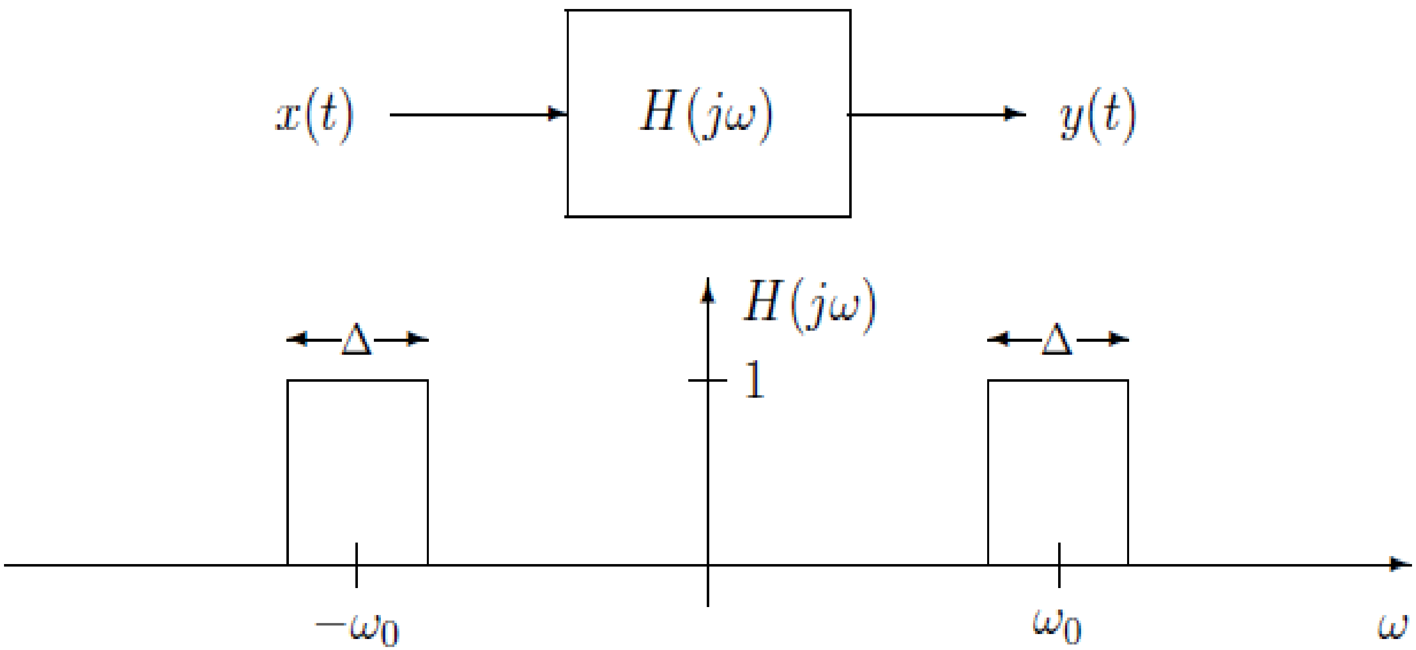
# Signal power and power spectral density, properties of power spectral density

Motivated by situations in which $x(t)$ is the voltage across (or current through) a unit resistor, we refer to $x^2(t)$ as the *instantaneous power* in the signal $x(t)$. When $x(t)$ is WSS, the *expected* instantaneous power is given by

$$E[x^2(t)] = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(j\omega)\, d\omega \ ,$$

where $S_{xx}(j\omega)$ is the CTFT of the autocorrelation function $R_{xx}(\tau)$. Furthermore, when $x(t)$ is ergodic in correlation, so that time averages and ensemble averages are equal in correlation computations, then (10.1) also represents the time-average power in any ensemble member. Note that since $R_{xx}(\tau) = R_{xx}(-\tau)$, we know $S_{xx}(j\omega)$ is always *real* and *even* in $\omega$; a simpler notation such as $P_{xx}(\omega)$ might therefore have been more appropriate for it, but we shall stick to $S_{xx}(j\omega)$ to avoid a proliferation of notational conventions, and to keep apparent the fact that this quantity is the Fourier transform of $R_{xx}(\tau)$.

The integral above suggests that we might be able to consider the expected (instantaneous) power (or, assuming the process is ergodic, the time-average power) in a frequency band of width $d\omega$ to be given by $(1/2\pi)S_{xx}(j\omega)\,d\omega$. To examine this thought further, consider extracting a band of frequency components of $x(t)$ by passing $x(t)$ through an ideal bandpass filter, shown in Figure



Because of the way we are obtaining $y(t)$ from $x(t)$, the expected power in the output $y(t)$ can be interpreted as the expected power that $x(t)$ has in the selected passband. Using the fact that

$$S_{yy}(j\omega) = |H(j\omega)|^2 S_{xx}(j\omega) \, ,$$

we see that this expected power can be computed as

$$E\{y^2(t)\} = R_{yy}(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{yy}(j\omega)\, d\omega = \frac{1}{2\pi} \int_{\text{passband}} S_{xx}(j\omega)\, d\omega \, .$$

Thus

$$\frac{1}{2\pi} \int_{\text{passband}} S_{xx}(j\omega)\, d\omega$$

is indeed the expected power of $x(t)$ in the passband. It is therefore reasonable to call $S_{xx}(j\omega)$ the power spectral density (PSD) of $x(t)$. Note that the instantaneous power of $y(t)$, and hence the expected instantaneous power $E[y^2(t)]$, is always nonnegative, no matter how narrow the passband, It follows that, in addition to being real and even in $\omega$, the PSD is always nonnegative, $S_{xx}(j\omega) \geq 0$ for all $\omega$. While the PSD $S_{xx}(j\omega)$ is the Fourier transform of the autocorrelation function, it is useful to have a name for the *Laplace* transform of the autocorrelation function; we shall refer to $S_{xx}(s)$ as the *complex* PSD.

Exactly parallel results apply for the DT case, leading to the conclusion that $S_{xx}(e^{j\Omega})$ is the power spectral density of $x[n]$.